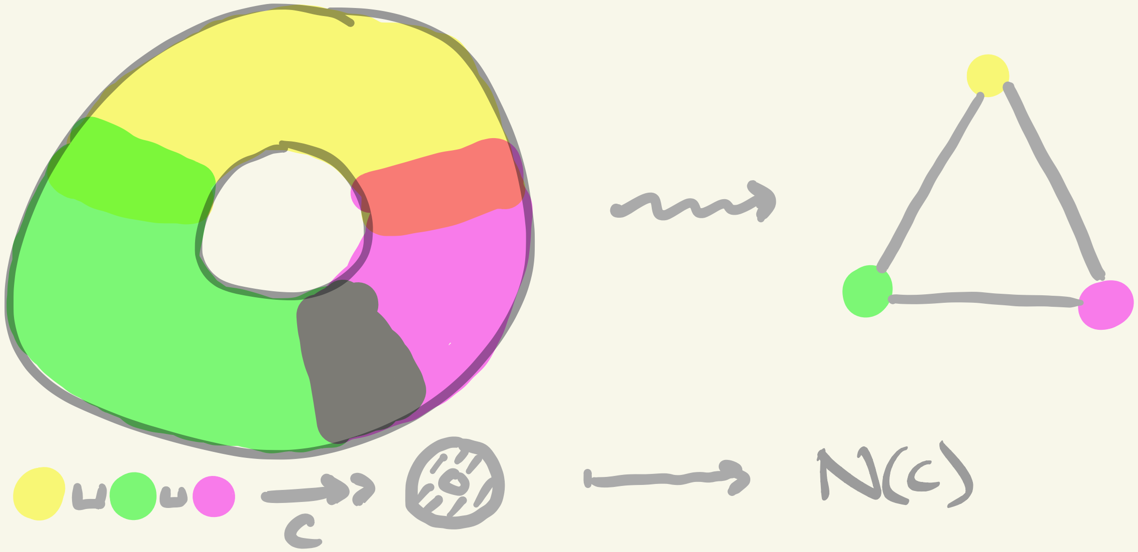
A Modal Proof of the Nerve Theorem

Simplicial arguments in cohesive homotopy type theory



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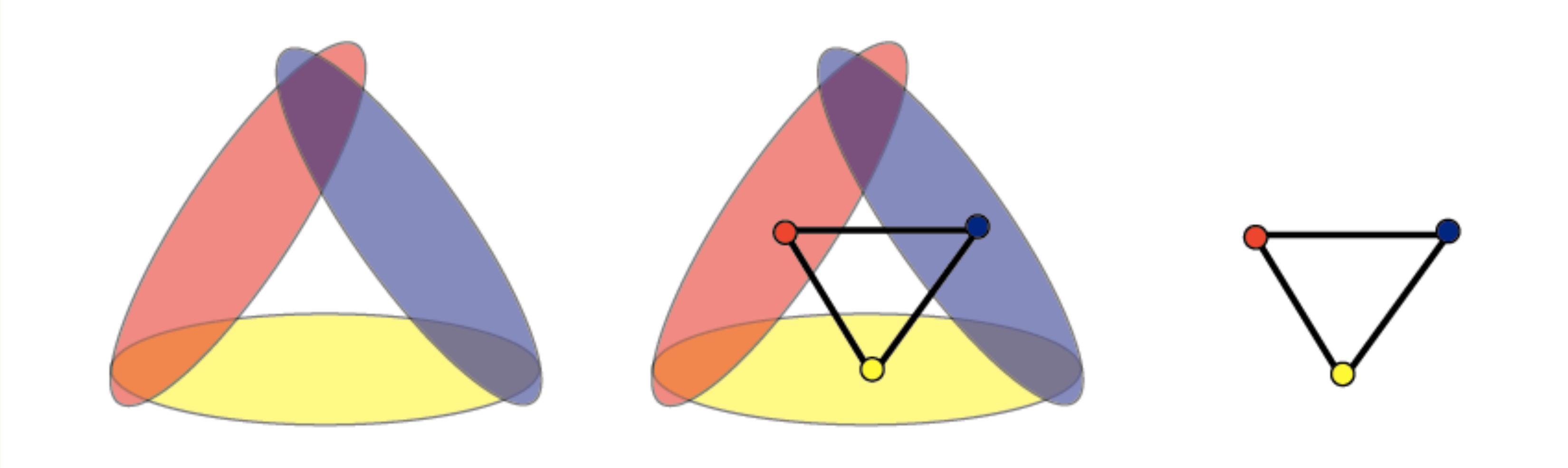
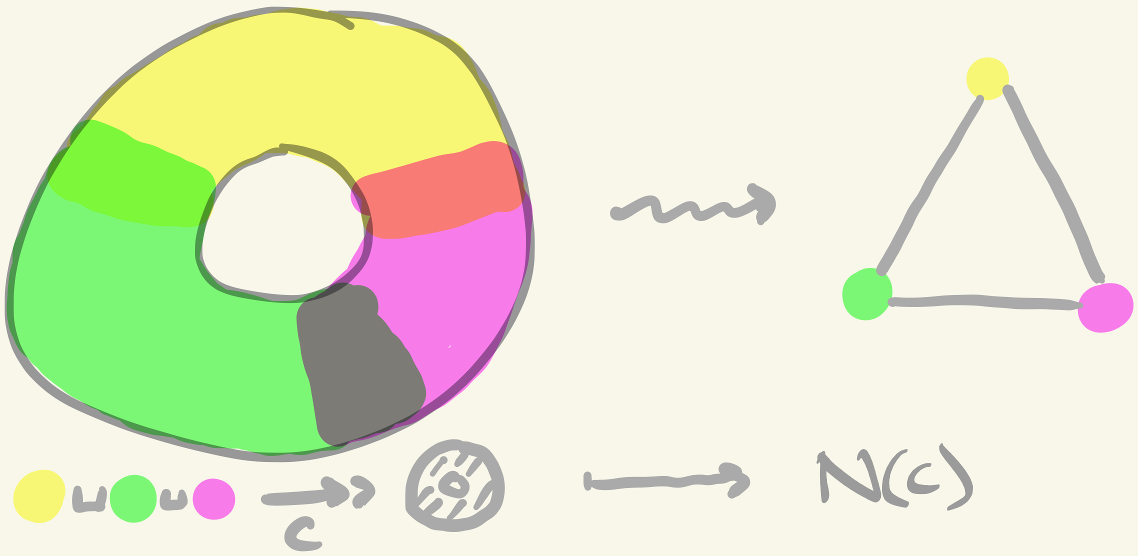
The Nerve Theorem



The (Alexandrov) nerve of an open cover $\{U_i\}_{i \in I}$ of M is the simplicial complex on the set I w/ an n-simplex $\{i_1, \ldots i_{n+1}\}$ whenever $U_{i_1} \cap \cdots \cap U_{i_n}$ is inhabited

Nerve Theorem (Leray): If $\{U_i\}_{i \in I}$ is good, and M is paracpt.
i.e. $U_{i_1} \cap \cdots \cap U_{i_n}$ is contractible whenever it is inhabited.
Then $N\{U_i\}_{i \in I}$ presents the homotopy type of M

The nerve theorem is about computing the homotopy type of a space.



So:

① What is a space?

② What is a homotopy type?

Spaces

Homotopy Types

These are two different things

There are many cts. maps $\mathbb{R} \to \mathbb{R}$, but only one up to homotopy

Spaces | Homotopy Types

$$\mathbb{R} \longmapsto *$$

$$S^n \longmapsto S^n$$

$$\mathbb{C}P^\infty \longmapsto B^2\mathbb{Z}$$

These are two different things

Taking the homotopy type of a space is an operation

What is a space?

§ 1. Magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisations. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness; the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Notions whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming notions whose specialisations make up a *continuous* manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in the higher mathematic.

On the Hypotheses which lie at the Bases of Geometry.

Bernhard Riemann

Translated by William Kingdon Clifford

What is a space?

- A space of possibilities ("points"):
  - possible locations or directions,
  - possible events,
  - possible parameters,
  - possible objects of a given sort.
- A space is a notion which admits many possible specializations.
- The points of spaces cohere in that they may fail to be separated by discrete and definite measures.

  E.g.: Any cts function $f : \mathbb{R} \to \{0, 1\}$ is necessarily constant.
- Conversely, discreteness of a space may be <u>defined</u> by those spaces it fails to separate.

Sheaves and Toposes

- Riemann's continuous manifoldnesses are very similar to Sheaves on the category of Euclidean spaces and smooth maps

$$\mathcal{X} \in Sh(Euc, \overset{\text{Good}}{\text{Open Covers}})$$

$\mathbb{R}^0 = * \longmapsto \mathcal{X}(*) =$ "Set of individual specialization of $\mathcal{X}$"

$\mathbb{R} \longmapsto \mathcal{X}(\mathbb{R}) =$ "Set of smooth paths in $\mathcal{X}$"

- $\mathcal{X}$ is Discrete when it "has no smooth paths between points"

$$\mathcal{X}(\mathbb{R}^n) \xrightarrow{\sim} \mathcal{X}(*)$$

or equivalently when "every path is constant"

$$\mathcal{X} \xrightarrow[\text{const}]{\sim} \mathcal{X}^{\mathbb{R}}$$

Sheaves and Toposes

- Similarly, simplicial sets are (pre-)sheaves on Δ

$$X \in \mathrm{Set}^{\Delta^{op}} \qquad [n] \longmapsto X_n = \text{"Set of } n\text{-simplices"}$$

and X is 0-skeletal when $X \xrightarrow[\text{Const}]{\sim} X^{\Delta[1]}$

- Both are examples of Lawvere's Axiomatic Cohesion.

$$\begin{tikzcd}[row sep=huge] \mathrm{Sh}(\mathrm{Euc}, \text{Good Open Covers}) \arrow[d, shift right=9, "\pi_0"', "\text{pieces}"] \arrow[d, shift right=3, "\text{points}"] \\ \mathrm{Set} \arrow[u, hook, shift right=6, "\text{discrete}"'] \arrow[u, hook, shift left=0, "\text{codiscrete}"'] \end{tikzcd} \qquad \begin{tikzcd}[row sep=huge] \mathrm{Set}^{\Delta^{op}} \arrow[d, shift right=9, "\mathrm{colim}"', "\text{pieces}"] \arrow[d, shift right=3, "\text{points}"] \\ \mathrm{Set} \arrow[u, hook, shift right=6, "0\text{-skeletal}"'] \arrow[u, hook, shift left=0, "0\text{-coskeletal}"'] \end{tikzcd}$$

Commuting Cohesions

$$\begin{tikzcd}
 & \mathrm{Sh}(\mathrm{Euc}, \text{Good open covers})^{\Delta^{op}} \arrow[dl, shift right=3] \arrow[dr, shift right=3] & \\
\mathrm{Sh}(\mathrm{Euc}, \text{Good open covers}) \arrow[ur, shift right=1] \arrow[ur, shift left=3] \arrow[dr, shift right=3] & & \mathrm{Set}^{\Delta^{op}} \arrow[ul, shift right=1] \arrow[ul, shift left=3] \arrow[dl, shift right=3] \\
 & \mathrm{Set} \arrow[ul, shift right=1] \arrow[ul, shift left=3] \arrow[ur, shift right=1] \arrow[ur, shift left=3] &
\end{tikzcd}$$

- The Čech nerve of an open cover $\coprod_{i\in I} U_i \xrightarrow{c} M$ is a simplicial space:

$$\check{C}(c)_n := \coprod_{\vec{i}\in I^{n+1}} \left(U_{i_1} \cap \cdots \cap U_{i_{n+1}}\right) = \lim \left(\begin{array}{c} \overbrace{\coprod_{i\in I} U_i \qquad \cdots \qquad \coprod_{i\in I} U_i}^{(n+1)\text{-times}} \\ \searrow^{c} \qquad \cdots \qquad \swarrow^{c} \\ M \end{array} \right)$$

What is a homotopy type?

§ 1. Magnitude-notions are only possible where there is an antecedent general notion which admits of different specialisations. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness; the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Notions whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a notion in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming notions whose specialisations make up a *continuous* manifoldness, that the only simple notions whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these notions occur first in the higher mathematic.

On the Hypotheses which lie at the Bases of Geometry.

Bernhard Riemann

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What is a homotopy type?

$$\boxed{\text{A homotopy type is an } \infty\text{-groupoid}}$$

- An ∞-groupoid X consists of
  - a notion of objects
  - for any objects x and y, an ∞-groupoid
    $$X(x,y)$$
    of equivalences between x and y
- \+ Compositions and coherences...

- E.g. the homotopy type $\int X$ of a topological space X is
  - The notion of a point in X, where
  - Two points x and y are identified by
    $$(\int X)(x,y) := \int \mathrm{Path}_X(x,y)$$
    the homotopy type of the path space.

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Discrete!

- E.g. the homotopy type $\int X$ of a topological space X is
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There are many different kinds of spaces and stacks:

- Topological Spaces
- Manifolds ($C^0 \cdots C^\infty$)
- Schemes
- Simplicial Sets
- Orbispaces
- Orbifolds and Lie Groupoids
- Deligne-Mumford Stacks
- Simplicial ∞-groupoids

These are all **sheaves of homotopy types** on various sites

$$\mathcal{E} = \{\text{Sheaves}\} \text{ forms an } \boldsymbol{\infty\text{-topos}}$$

Eg: $$\{\text{Manifolds}\} \cup \left\{\begin{matrix}\text{Orbifolds} \\ \text{and} \\ \text{Lie Groupoids}\end{matrix}\right\} \longrightarrow Sh_\infty\left(\mathcal{E}uc, \begin{matrix}\text{Good} \\ \text{Open Covers}\end{matrix}\right)$$

$$\begin{tikzcd}[column sep=large]
\mathrm{Sh}_\infty(\mathcal{E}uc, \text{Good open covers}) \arrow[loop above, "\int\ \text{``shape''}"] \arrow[loop above, "\flat\ \text{``flat''}"] \arrow[loop above, "\sharp\ \text{``sharp''}"] \arrow[d, shift right=6, "\text{homotopy type}"] \arrow[d, shift left=2, "\text{points}"] \\
\infty\text{-Groupoids} \arrow[u, hook, shift right=2, "\text{discrete}"] \arrow[u, hook, shift left=6, "\text{codiscrete}"]
\end{tikzcd}$$

- In higher toposes, cohomology theories become representable.
- In cohesive higher toposes (Schreiber), these become differential cohomology theories.

$$\left\{ \int X \xrightarrow{c} \mathbf{B}^n_\nabla U(1) \right\} \simeq \left\{ X \xrightarrow{c} \flat\, \mathbf{B}^n_\nabla U(1) \right\}$$

local systems

"classifying stack"

bundles w/ flat connection

Homotopy Type Theory is

- a logical system for working directly with sheaves of homotopy types.
- a standalone foundation of mathematics
  - Types A of mathematical objects
  - Elements $\underbrace{a : A \text{ of a given type. "} a \text{ is an}}$ A"
  - Variable Elements $x^2+1 : \mathbb{R}$ (given that $x : \mathbb{R}$)

$$\underbrace{x : \mathbb{R}}_{\text{"Context"}} \vdash x^2+1 : \mathbb{R}$$

  - Variable types $M : \mathrm{Manifold},\ p : M \vdash T_pM : \mathrm{Vect}_{\mathbb{R}}$

$\mathbb{N}$ is the type of natural numbers
$\mathbb{R}$ is the type of real numbers
Set is the type of sets
$\mathrm{Vect}_{\mathbb{R}}$ is the type of real vector spaces
Type is the type of types.

[$x : A \vdash b(x) : B(x)$ means "$b(x)$ is a $B(x)$, given that x is an A"]

Pair Types:

$TM :\equiv (p : M) \times T_p M$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \times B(x) \qquad A \times B$$

is the type of pairs (a, b) with $a : A$ and $b : B(a)$.

Function Types:

$Vec(M) :\equiv (p : M) \to T_p M$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \to B(x) \qquad A \to B$$

is the type of functions $x \mapsto f(x)$ where $x : A \vdash f(x) : B(x)$

Types of Identifications:

- If x and y are of type A, then

$$x =_A y$$ is the type of ways to identify x with y as elements of A.

E.g.

- In $\mathrm{Vect}_{\mathbb{R}}$, $e : T_pM = \mathbb{R}^n$ is a linear isomorphism.
- In Manifold, $e : M = N$ is a diffeomorphism.
- In Type, $e : A = B$ is an equivalence.
- In $\mathbb{N}$, $n = m$ has a unique element if and only if n equals m.

"Univalence Axiom" of Voevodsky

Types of Structures

$$
\mathbf{B}U(1) \equiv \begin{cases}
(L : \mathbf{Type}) \times \mathsf{isSet}(L) \\
\qquad \times (+ : L \times L \to L) \times (0 : L) \\
\qquad \times (\cdot : \mathbb{C} \times L \to L) \\
\qquad \times (\mathsf{assoc} : (a, b, c : L) \to (a + b) + c = a + (b + c)) \\
\qquad \times (\mathsf{zero} : (a : L) \to a + 0 = a) \\
\qquad \times (\mathsf{comm} : (a, b : L) \to a + b = b + a) \\
\qquad \times (\mathsf{unit} : (a : L) \to 1 \cdot a = a) \\
\qquad \times (\mathsf{scalarassoc} : (u, v : \mathbb{C}) \to (a : L) \to u \cdot (v \cdot a) = (uv) \cdot a) \\
\qquad \times (\mathsf{scalarzero} : (a : L) \to 0 \cdot a = 0) \\
\qquad \times (\mathsf{dim} : \exists((b : L) \times \neg(b = 0) \times ((a : L) \to (c : \mathbb{C}) \times (a = c \cdot b)))) \\
\qquad \times (\langle -, - \rangle : L \times L \to \mathbb{C}) \\
\qquad \times (\mathsf{leftlinear} : (c, d : \mathbb{C}) \to (a, b, z : L) \to (\langle ca + db, z \rangle = c\langle a, z \rangle + d\langle b, z \rangle)) \\
\qquad \times (\mathsf{sesqui} : (a, b : L) \to \langle a, b \rangle = \overline{\langle b, a \rangle}) \\
\qquad \times (\mathsf{positive} : (a : L) \to \langle a, a \rangle \geq 0) \\
\qquad \times (\mathsf{definite} : (a : L) \to ((b : L) \to \langle a, b \rangle = 0) \to (a = 0))
\end{cases}
$$

Dictionary (Shulman, Lumsdaine, Kapulkin, Voevodsky, et al.)

Homotopy Type Theory	Sheaves of homotopy types
Type of object	Sheaf of homotopy types in $\mathcal{E}$
$x : A \vdash B(x) : \mathrm{Type}$	$B \xrightarrow{\pi} A$ in $\mathcal{E}/A$
$x : A \vdash b(x) : B(x)$	$\begin{tikzcd} A \arrow[r, "b"] \arrow[dr, equal] & B \arrow[d, "\pi"] \\ & A \end{tikzcd}$ in $\mathcal{E}/A$
$(x : A) \times B(x)$	$\begin{tikzcd} B \arrow[d] \arrow[dr, dashed] & \\ A \arrow[r] & * \end{tikzcd}$ along $\mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E}/*$
$(x : A) \to B(x)$	$\left\{ \begin{tikzcd} B \arrow[dr] & \\ & A \arrow[ul, dashed, bend left] \end{tikzcd} \right\}$ along $\mathcal{E}/A \xrightarrow{\Pi_A} \mathcal{E}/*$
$x, y : A \vdash (x = y) : \mathrm{Type}$	$A \xrightarrow{\Delta} A \times A$ The diagonal in $\mathcal{E}/A \times A$

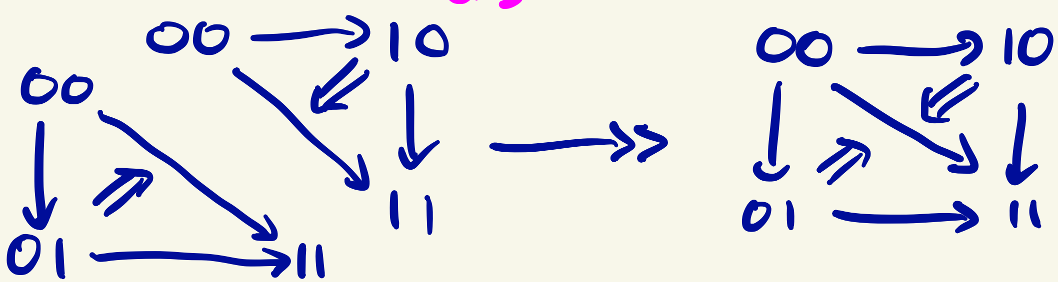
Simplicial Cohesion $\quad re \dashv sk_0 \dashv csk_0$

- Dip into simplicial arguments by adding a simplicial focus

Ax: $\Delta[1]$ is a total order with distinct top 1 and bottom 0.

The axiom total : $\forall i,j : \Delta[1].\ i \le j \vee j \le i$ means that

$$\Delta[1]\times\Delta[1] = \underbrace{\{(i,j) : \Delta[1]^2 \mid i \le j\}}_{\Delta[2]} \cup \{(i,j) : \Delta[1]^2 \mid j \le i\}$$



Def: $\Delta[n] :\equiv \{(x_1, \ldots, x_n) : \Delta[1]^n \mid x_1 \le x_2 \le \cdots \le x_n\}$

If X is simplicially crisp, then $X_n :\equiv sk_0(\Delta[n] \to X)$

Simplicial Cohesion $\quad re \dashv sk_0 \dashv csk_0$

- sk_0 is the 0-skeleton $\quad (sk_0 X)_n = X_0$
- csk_0 is the 0-coskeleton $\quad (csk_0 X)_n = X^{n+1}$
- re is the realization $\quad (re\, X)_n = \mathrm{Colim}(X : \Delta^{op} \to \mathcal{U}_0)$

$X \xrightarrow{(-)^{re}} reX$ is an equivalence iff $X \xrightarrow{\text{const}} X^{\Delta[1]}$ is an equivalence iff $sk_0 X \xrightarrow{(-)_{sk_0}} X$ is an equivalence

These are <u>modalities</u>

re is reflection into the 0-skeletal types.

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d] & Z \\ reX \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

when $sk_0 Z \xrightarrow{\sim} Z$

csk_0 is reflection into the 0-coskeletal types

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d] & Z \\ csk_0 X \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

when $Z \xrightarrow{\sim} csk_0 Z$

Constructions with the cohesive modalities:

Thm (M.-Riley): Let $f : X \to Y$ be a simplicially crisp map between 0-skeletal types. Then its csk_0-modal factor

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, dashed, "\mathrm{csk}_0\text{-equiv}"'] & & Y \\
& (y:Y) \times \mathrm{csk}_0((x:X) \times (fx = y)) \arrow[ur, dashed, "\mathrm{csk}_0\text{-modal}"'] &
\end{tikzcd}$$

is the Čech nerve $\check{C}(f)$ of f.

Proof:

$$\begin{aligned}
&\mathrm{sk}_0(\Delta[n] \to (y:Y) \times \mathrm{csk}_0((x:X) \times (fx = y))) \\
&\simeq \mathrm{sk}_0((\sigma : \Delta[n] \to Y) \times ((i : \Delta[n]) \to \mathrm{csk}_0((x:X) \times (fx = \sigma i)))) \\
&\simeq \mathrm{sk}_0((y:Y) \times (\Delta[n] \to \mathrm{csk}_0((x:X) \times (fx = y)))) \\
&\simeq (y:Y) \times \mathrm{sk}_0(\Delta[n] \to \mathrm{csk}_0((x:X) \times (fx = y))) \\
&\simeq (y:Y) \times \mathrm{sk}_0([n] \to (x:X) \times (fx = y)) \\
&\simeq (y:Y) \times ((x:X) \times (fx = y))^{n+1}
\end{aligned}$$

$$\begin{array}{cc}
\vdots & \\
X \times_Y X \times_Y X & 2 \\
\downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow & \\
X \times_Y X & 1 \\
\downarrow\ \uparrow\ \downarrow & \\
X & 0
\end{array}$$

Cor (M.-Riley): ... and $\mathrm{re}\,\check{C}(f) \simeq \mathrm{im}\, f$. (b/c $\mathrm{re}\,\mathrm{csk}_0 X = \exists X$)

Nerve of an open cover:

- If $\{U_i\}_{i:I}$ is a cover of M, then

$$\coprod_{i:I} U_i \overset{c}{\twoheadrightarrow} M$$

is a surjection, so we have (effective epimorphism)

$$M = \mathrm{im}(c) = \mathrm{re}\,\check{C}(c)$$

- The **nerve** of the cover is the simplicial set with an n-simplex for every inhabited $(n+1)$-fold intersection

$$\begin{tikzcd}
\coprod_{i:I} U_i \arrow[r, "c"] \arrow[d] & \check{C}(c) \arrow[r] \arrow[d, "\pi"] & M \arrow[d] \\
I \arrow[r] & \mathrm{csk}_0(I) \arrow[r] & *
\end{tikzcd}$$

ie

$$N(c) := \mathrm{im}(\pi)$$

$$N(c)_n \simeq \{i : I^{n+1} \mid \exists (U_{i_1} \cap \dots \cap U_{i_{n+1}})\}$$

Good open covers

- The nerve of the cover is the simplicial set with an n-simplex for every inhabited $(n+1)$-fold intersection

$$\begin{tikzcd} \bigsqcup_{i:I} U_i \arrow[r, "c"] \arrow[d] & \check{C}(c) \arrow[r] \arrow[d, "\pi"] & M \arrow[d] \\ I \arrow[r] & \operatorname{csk}_0(I) \arrow[r] & * \end{tikzcd}$$

ie

$$N(c) := \operatorname{im}(\pi)$$

$$N(c)_n \simeq \{ i : I^{n+1} \mid \exists (U_{i_1} \cap \cdots \cap U_{i_{n+1}}) \}$$

- A cover is good if $U_{i_1} \cap \cdots \cap U_{i_{n+1}}$ is contractible whenever it is inhabited

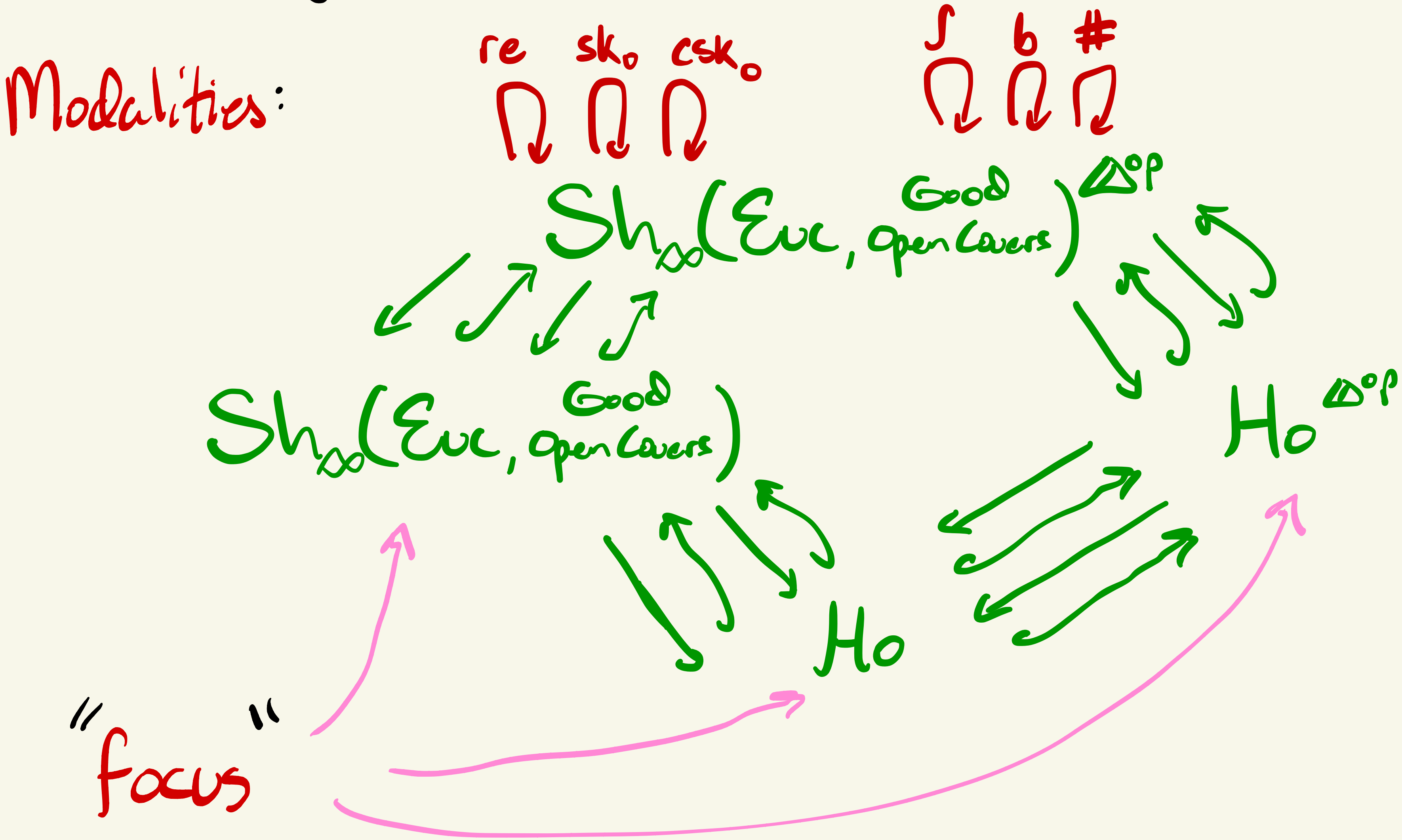
ie: $\int (U_{i_1} \cap \cdots U_{i_{n+1}}) \xrightarrow{\sim} \exists (U_{i_1} \cap \cdots \cap U_{i_{n+1}})$

Lemma:

c is good iff its nerve is the homotopy type of its Čech nerve.

$$\int \check{C}(c) \xrightarrow{\sim} N(c)$$

Commuting Cohesions



Type Theory with Commuting Cohesions (M.-Riley)
(Extending Shulman's Cohesive HoTT.)

- Free variables annotated w/ a focus

$$x :_{\heartsuit} A \vdash b : B$$

A variable in focus $\heartsuit$ is said to be $\heartsuit$-crisp.

- Elements vary "$\heartsuit$-discontinuously" in their $\heartsuit$-crisp variables.

E.g. $x :_{\mathrm{sk}_0} A,\ y :_{\flat} B,\ z :_{\flat\mathrm{sk}_0} C \vdash f : D$

is an element f of the simplicial differential stack D which depends smoothly on a point x of A and simplicially but discontinuously on y in B, and discontinuously on a point z of C.

Type Theory with Commuting Cohesions (M.-Riley)

- For any focus ♥, we have (co)modalities

$$\flat_\heartsuit \quad \text{and} \quad \sharp_\heartsuit$$

representing ♥-crisp variables on the left and right.

$\flat_\heartsuit$:

$$\flat\text{-FORM}\ \frac{\heartsuit \backslash \Gamma \vdash A\ \text{type}}{\Gamma \vdash \flat_\heartsuit A\ \text{type}}$$

$$\flat\text{-INTRO}\ \frac{\heartsuit \backslash \Gamma \vdash M : A}{\Gamma \vdash M^{\flat\heartsuit} : \flat_\heartsuit A} \qquad \flat\text{-ELIM}\ \frac{\clubsuit\heartsuit \backslash \Gamma \vdash A\ \text{type} \quad \Gamma, x :_\clubsuit \flat_\heartsuit A \vdash C\ \text{type} \quad \clubsuit \backslash \Gamma \vdash M : \flat_\heartsuit A \quad \Gamma, u :_{\clubsuit\heartsuit} A \vdash N : C[u^{\flat\heartsuit}/x]}{\Gamma \vdash (\text{let } u^{\flat\heartsuit} := M \text{ in } N) : C[M/x]}$$

$$\flat\text{-BETA}\ \frac{\clubsuit\heartsuit \backslash \Gamma \vdash A\ \text{type} \quad \Gamma, x :_\clubsuit \flat_\heartsuit A \vdash C\ \text{type} \quad \clubsuit\heartsuit \backslash \Gamma \vdash K : A \quad \Gamma, u :_{\clubsuit\heartsuit} A \vdash N : C[u^{\flat\heartsuit}/x]}{\Gamma \vdash (\text{let } u^{\flat\heartsuit} := K^{\flat\heartsuit} \text{ in } N) \equiv N[K/u] \ : C[K^{\flat\heartsuit}/x]}$$

E.g. $\flat$ Sk_0

$\sharp_\heartsuit$:

$$\sharp\text{-FORM}\ \frac{\heartsuit \Gamma \vdash A\ \text{type}}{\Gamma \vdash \sharp_\heartsuit A\ \text{type}} \qquad \sharp\text{-INTRO}\ \frac{\heartsuit \Gamma \vdash M : A}{\Gamma \vdash M^{\sharp\heartsuit} : \sharp_\heartsuit A} \qquad \sharp\text{-ELIM}\ \frac{\heartsuit \backslash \Gamma \vdash N : \sharp_\heartsuit A}{\Gamma \vdash N_{\sharp_\heartsuit} : A}$$

$$\sharp\text{-BETA}\ \frac{\heartsuit \backslash \Gamma \vdash M : A}{\Gamma \vdash (M^{\sharp\heartsuit})_{\sharp_\heartsuit} \equiv M : A} \qquad \sharp\text{-ETA}\ \frac{\Gamma \vdash N : \sharp_\heartsuit A}{\Gamma \vdash N \equiv (N_{\sharp_\heartsuit})^{\sharp\heartsuit} : \sharp_\heartsuit A}$$

E.g. $\sharp$ cSk_0

Commuting Cohesions

- $\mathbb{R}$ is 0-skeletal and $\Delta[1]$ is discrete

$$\mathbb{R} \xrightarrow{\sim} \mathbb{R}^{\Delta[1]} \qquad \Delta[1] \xrightarrow{\sim} \Delta[1]^{\mathbb{R}}$$

So their localizations commute: $\int \mathrm{re} = \mathrm{re} \int$

- $\int$ is computed pointwise, so also: $\mathrm{sk}_0 \int = \int \mathrm{sk}_0$ $\left(\begin{array}{c}\text{also} \\ \int \exists = \exists\end{array}\right)$

and: $\mathrm{csk}_0 \int = \int \mathrm{csk}_0$

Thm (M.-Riley): If $c : \coprod_{i:I} U_i \twoheadrightarrow M$ is a good cover, then

$$\int M \simeq \mathrm{re}\, N(c)$$

Proof:

$$\int M \simeq \int \mathrm{re}\, \check{C}(c) \simeq \mathrm{re} \int \check{C}(c) \simeq \mathrm{re}\, N(c)$$

Modal Nerve Theorem

- Works for many modalities that commute w/ the simplicial ones

$$L\,re \overset{!}{=} re\,L \qquad csk_0 L = L\, csk_0$$

$$sk_0 L = L\, sk_0 \qquad L\exists = \exists$$

Just so that "L-good" means $L(U_{i_0} \cap \cdots \cap U_{i_{n+1}}) = \exists(U_{i_0} \cap \cdots \cap U_{i_{n+1}})$

E.g. $L = L_0^{\Delta^{op}}$ for $L_0 : \mathcal{E} \to \mathcal{E}$ st $L\exists = \exists$ and $L \operatorname{colim}_{\Delta^{op}} = \operatorname{colim}_{\Delta^{op}} L$.

Thm (M.-Riley): If $c : \coprod_{i:I} U_i \twoheadrightarrow M$ is a L-good cover, (0-skeletal) and I is L-modal, 0-skeletal, and simplicially crisp then:

$$LM \simeq L\,re\,\check{C}(c) \simeq re\,L\,\check{C}(c) \simeq re\,N(c)$$

Thank You!

references:

Commuting Cohesions

David Jaz Myers   Mitchell Riley

A TYPE THEORY FOR SYNTHETIC ∞-CATEGORIES

EMILY RIEHL AND MICHAEL SHULMAN

BROUWER'S FIXED-POINT THEOREM IN REAL-COHESIVE HOMOTOPY TYPE THEORY

MICHAEL SHULMAN

Urs Schreiber
Differential cohomology in a cohesive ∞-topos
v1: arXiv:1310.7930
v2: ***pdf***

Propositions as types

- $\ulcorner X$ has a unique element$\urcorner :\equiv (c:X)\times((x:X)\to(x=c))$
  "$\exists! X$"
- For $f : X\to Y$,
  $\ulcorner f$ is an equivalence$\urcorner :\equiv (y:Y)\to \ulcorner (x:X)\times(fx=y)$ has a unique element$\urcorner$
  "$\forall y:Y, \exists! x:X, fx=y$"
- $\ulcorner P$ is a proposition$\urcorner :\equiv (x,y:P)\to\ulcorner(x=y)$ has a unique element$\urcorner$
  "P has at most one element"

To prove P is to give an element of it. — "the fact that P" $: P$

$P\Rightarrow Q :\equiv P\to Q$, $\quad P\wedge Q :\equiv P\times Q$, $\quad \forall x:X.\,P(x) :\equiv (x:X)\to P(x)$

Propositional truncation:

For any type X, a proposition $\exists X$ and $(-)^{\exists} : X\to\exists X$,
initial for such maps

$$\begin{tikzcd}
X \arrow[r, "\forall"] \arrow[d, "(-)^{\exists}"'] & P \\
\exists X \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

Types of Structures

- $\ulcorner X \text{ is a set} \urcorner :\equiv (x, y : X) \to \ulcorner (x = y) \text{ is a proposition} \urcorner$

$$\{x : X \mid P(x)\} :\equiv (x : X) \times P(x)$$

- $\text{Monoid} :\equiv (M : \text{Type}) \times (\cdot : M \times M \to M) \times (1 : M)$ Structure

$$\begin{aligned} &\times \ulcorner M \text{ is a set} \urcorner \\ &\times ((x : M) \to (1 \cdot x = x)) \\ &\times ((x : M) \to (x \cdot 1 = x)) \\ &\times ((x\, y\, z : M) \to ((x \cdot y) \cdot z = x \cdot (y \cdot z)) \end{aligned}$$

Properties

- For G a group,

$$\begin{aligned} \text{Tors}_G :\equiv (T : \text{Type}) &\times (\alpha : G \times T \to T) \\ &\times \ulcorner T \text{ is a set} \urcorner \\ &\times \forall t : T,\ \alpha(1, t) = t \\ &\times \forall t : T,\ g, h : G,\ \alpha(gh, t) = \alpha(g, \alpha(h, t)) \\ &\times \forall t, t' : T,\ \ulcorner \{g : G \mid \alpha(g, t) = t'\} \text{ has a unique element} \urcorner \\ &\times \exists T \end{aligned}$$