Topological Quantum Gates in Homotopy Type Theory

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

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3 2 1

Topological Quantum Gates

6 in

Homotopy Type Theory

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Gates Quantum Topological,
Type Theory Homotopy in

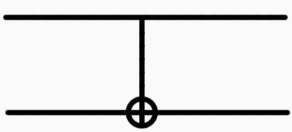
Computing

Classical

$\mathrm{Bit} = \{0\} + \{1\}$

(reversible) logic gate

$\mathrm{CNOT} : \mathrm{Bit} \times \mathrm{Bit} \to \mathrm{Bit} \times \mathrm{Bit}$



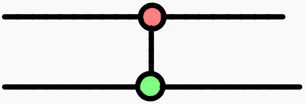
$$\begin{aligned} 0,0 &\longmapsto 0,0 \\ 0,1 &\longmapsto 0,1 \\ 1,0 &\longmapsto 1,1 \\ 1,1 &\longmapsto 1,0 \end{aligned}$$

Quantum

$\mathrm{Qbit} = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$

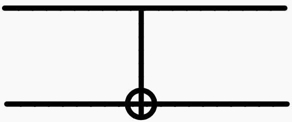
quantum gate

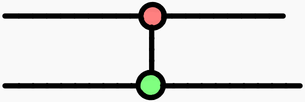
$\mathrm{CNOT} : \mathrm{Qbit} \otimes \mathrm{Qbit} \to \mathrm{Qbit} \otimes \mathrm{Qbit}$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Diagrams





Denotational Semantics

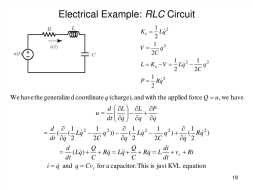
a function

CNOT : Bit × Bit → Bit × Bit

a unitary function

CNOT : Qbit ⊗ Qbit → Qbit ⊗ Qbit

Operational Semantics


Electrical Example: *RLC* Circuit

$K_e = \frac{1}{2}L\dot{q}^2$

$V = \frac{1}{2C}q^2$

$L = K_e - V = \frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2$

$P = \frac{1}{2}R\dot{q}^2$

We have the generalized coordinate q (charge), and with the applied force $Q = u$, we have

$$u = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial P}{\partial \dot{q}}$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2\right)\right) - \frac{\partial}{\partial q}\left(\frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2\right) + \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}R\dot{q}^2\right)$$

$$= \frac{d}{dt}(L\dot{q}) + \frac{Q}{C} + R\dot{q} = L\ddot{q} + \frac{Q}{C} + R\dot{q} = L\frac{di}{dt} + v_c + Ri$$

$i = \dot{q}$ and $q = Cv_c$ for a capacitor. This is just KVL equation

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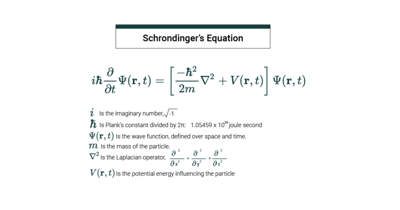

Schrondinger's Equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right]\Psi(\mathbf{r},t)$$

i Is the imaginary number, $\sqrt{-1}$

$\hbar$ Is Plank's constant divided by 2π: 1.05459 x 10^{34} joule second

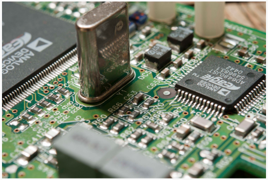
$\Psi(\mathbf{r},t)$ Is the wave function, defined over space and time.

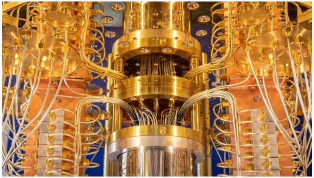
m Is the mass of the particle.

∇^2 Is the Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$V(\mathbf{r},t)$ Is the potential energy influencing the particle

Implementation





Quantum Mechanics

- A Hilbert space $\mathcal{H}_0$ of "states".
- A Hamiltonian operator $H : \mathcal{H}_0 \to \mathcal{H}_0$
- Dynamics given by Schrodinger's equation

$$\frac{d\varphi}{dt} = \frac{-i}{\hbar} H \varphi$$

- Solved to a one-parameter family of operators

$$U(t) : H_0 \longrightarrow \mathcal{H}_0 \qquad U(t) = e^{\frac{-i}{\hbar} Ht}$$

- This "implements" a gate G if $U(t_0)\varphi = G\varphi$.

Quantum Mechanics

- A Hilbert space $\mathcal{H}_0$ of "states".
- A Hamiltonian operator $H : \mathcal{H}_0 \to \mathcal{H}_0$
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What's this?

$$\frac{d\psi}{dt} = \frac{-i}{\hbar} H \psi$$

- Solved to a one-parameter family of operators

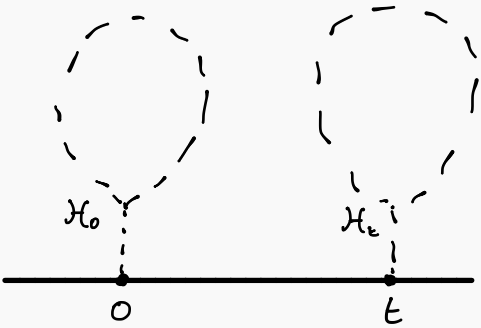
$$U(t) : \mathcal{H}_0 \to \mathcal{H}_0 \qquad U(t) = e^{\frac{-i}{\hbar} Ht}$$

- This "implements" a gate G if $U(t_0)\psi = G\psi$.

Quantum Mechanics

a Hilbert <u>bundle</u>

$$\begin{array}{c}\mathcal{H}\\ \big\downarrow\\ \mathbb{R}\end{array}$$



Quantum Mechanics

a Hilbert bundle

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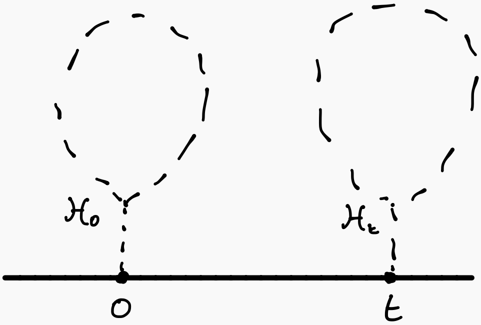


trivializable $\Longrightarrow$

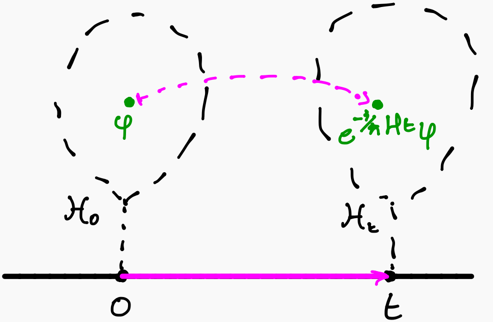
a connection ∇ is given by $H\,dt \in \Lambda^1(\mathbb{R}; \mathrm{End}(\mathcal{H}_0))$

the Hamiltonian!

Quantum Mechanics

a Hilbert bundle

$$\begin{tikzcd} \mathcal{H} \arrow[d] \\ \mathbb{R} \end{tikzcd}$$



trivializable $\Longrightarrow$

a connection ∇ is given by $H\,dt \in \Omega^1(\mathbb{R}; \mathrm{End}(\mathcal{H}_0))$

the Hamiltonian!

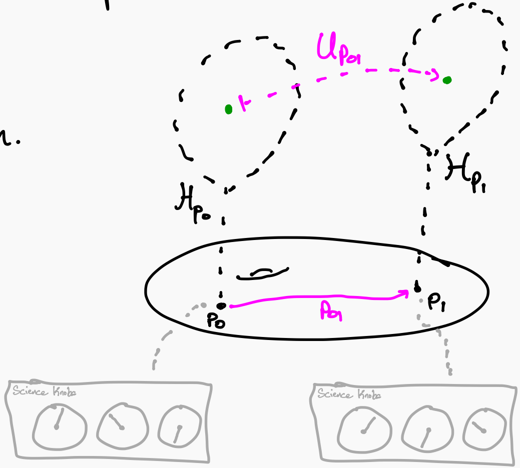
Solving Schrodinger's eqn. gives parallel transport! (if H is indep. of t)

Quantum Mechanics

Time isn't the only classical parameter...

A quantum system is a Hilbert bundle w/ connection.

$$\begin{array}{c} \mathcal{H} \\ \downarrow \\ P \end{array}, \quad \nabla$$

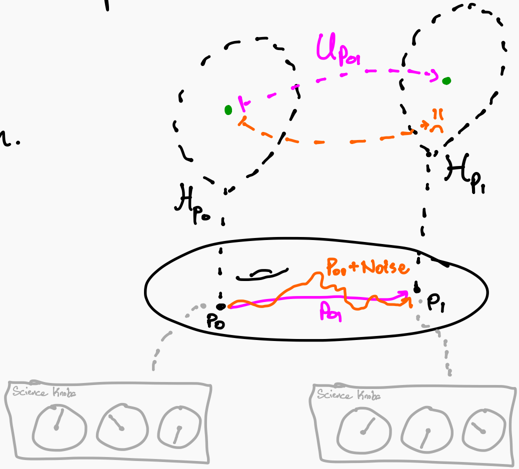


Quantum ~~Mechanics~~ Computing is hard

Time isn't the only classical parameter...

A quantum system is
a Hilbert bundle w/ connection.

$$\mathcal{H} \xrightarrow{\quad} P \ , \ \nabla$$



Highly susceptible to noise

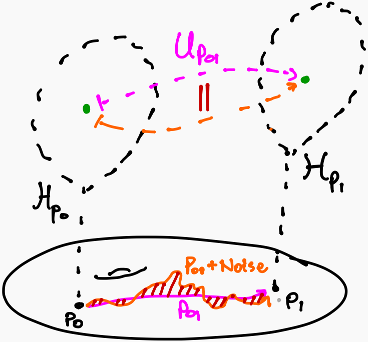
↳ Hard to scale to many Qbits (requiring more parameters)

Quantum ~~Mechanics~~ Computing is hard

Time isn't the only classical parameter...

A quantum system is a Hilbert bundle w/ connection.

$$\begin{array}{c}\mathcal{H}\\ \Big\downarrow \\ P\end{array}, \ \nabla$$



If the connection is flat then parallel transport is homotopy invariant.

Topological Quantum Systems

A quantum system $\begin{matrix}\mathcal{H}\\ \downarrow\\ P\end{matrix}, \nabla$ is topologically insulated if ∇ is flat.

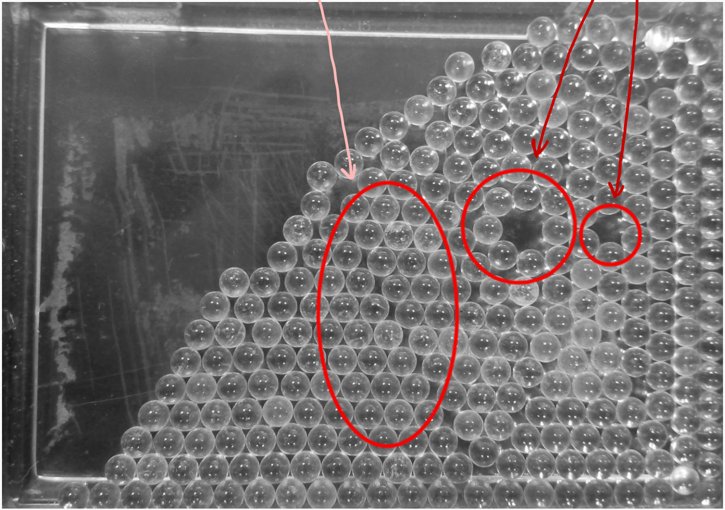
Since parallell transport is homotopy invariant, determined by a representation of $\pi_1 P$

$$\mathcal{H} : B\pi_1 P \longrightarrow \mathrm{Hilb}$$

Braiding Defects in Topological Quantum Materials

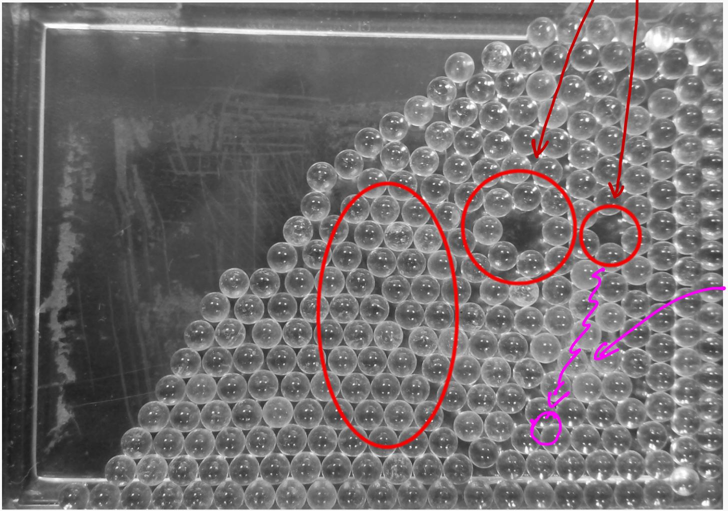
line defect

Point defects



Braiding Defects in Topological Quantum Materials

Point defects

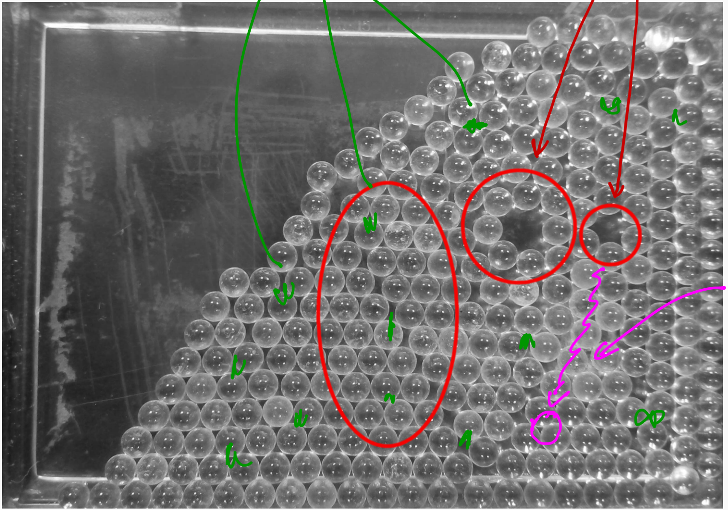


Moving defects

Braiding Defects in Topological Quantum Materials

anyonic particle excitations on the lattice

Point defects



Moving defects

Braiding Defects in Topological Quantum Materials

Paths in configuration space are braids

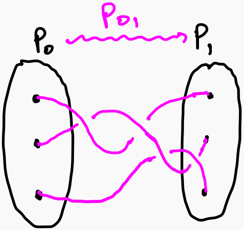
$$\left\{\begin{array}{c}\text{Conformal}\\ \text{Blocks}\end{array}\right\}$$

$$\downarrow$$

$$\mathrm{Conf}_d(\mathbb{R}^2)$$

Knizhnik-Zamolodchikov Connection

number of defects



Hypergeometric Integral Construction

[Sati-Schreiber, "anyonic defects" in TED K-theory]

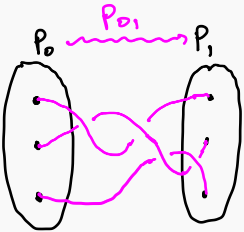
twisted cohomology

Configurations of n anyonic particles

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{Conformal}\\ \text{Blocks}\end{array}\right\} & & \underset{c\,\in\,\mathrm{Conf}_d}{\bigcup} H^n_\tau\big(\mathrm{Conf}_n(\mathbb{R}^2 - c);\, \mathbb{C}\big) \\
\big\downarrow & \simeq & \big\downarrow \\
\mathrm{Conf}_d(\mathbb{R}^2) & & \mathrm{Conf}_d(\mathbb{R}^2)
\end{array}
$$

, Knizhnik-Zamolodchikov Connection

, Gauss-Manin Connection



Hypergeometric Integral Construction

twisted cohomology

Configurations of n anyonic particles

$$\left\{\begin{array}{c}\text{Conformal}\\ \text{Blocks}\end{array}\right\} \longrightarrow \mathrm{Conf}_d(\mathbb{R}^2)\,, \quad \begin{array}{l}\text{Knizhnik}\\ \text{-Zamolodchikov}\\ \text{Connection}\end{array} \quad \simeq \quad \bigcup_{c \in \mathrm{Conf}_d} H^n_\tau\big(\mathrm{Conf}_n(\mathbb{R}^2 - c);\, \mathbb{C}\big) \longrightarrow \mathrm{Conf}_d(\mathbb{R}^2)\,, \quad \begin{array}{l}\text{Gauss}\\ \text{-Manin}\\ \text{Connection}\end{array}$$

Analysis

Geometry

Conformal Field Theory

<u>pure</u> Homotopy theory

Homological algebra

Representation theory

Hypergeometric Integral Construction

Since the Gauss-Manin connection is purely homotopical, we can use

homotopy type theory

to give an abstract **formal verification** (and even classical simulation!) of this realistic proposal for TQC.

Homotopy Type Theory is

- a logical system for working directly with homotopy types (spaces up to homotopy)
- a standalone foundation of mathematics
  - Types A of mathematical objects
  - Elements $a : A$ of a given type. "a is an A"
  - Variable Elements $x^2+1 : \mathbb{R}$ (given that $x : \mathbb{R}$)

$$\underbrace{x : \mathbb{R}}_{\text{"Context"}} \vdash x^2+1 : \mathbb{R}$$

  - Variable types $M : \mathrm{Manifold},\ p : M \vdash T_p M : \mathrm{Vect}_{\mathbb{R}}$

$\mathbb{N}$ is the type of natural numbers
$\mathbb{R}$ is the type of real numbers
Set is the type of sets
$\mathrm{Vect}_{\mathbb{R}}$ is the type of real vector spaces
Type is the type of types.

[$x : A \vdash b(x) : B(x)$ means "$b(x)$ is a $B(x)$, given that x is an A"]

Pair Types: $TM := (p : M) \times T_pM$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \times B(x) \qquad A \times B$$

is the type of pairs (a, b) with $a : A$ and $b : B(a)$.

Function Types: $\mathrm{Vec}(M) := (p : M) \to T_pM$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \to B(x) \qquad A \to B$$

is the type of functions $x \mapsto f(x)$ where $x : A \vdash f(x) : B(x)$

Paths and transport (Cubical Type Theory)

- We assume a primitive interval exotype $\mathbb{I}$ with endpoints $i0, i1 : \mathbb{I}$.

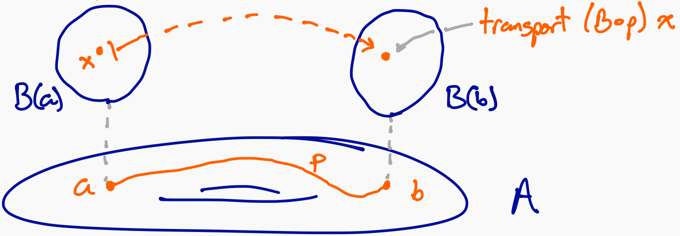
- A path $p : a \equiv b$ is a function $p : \mathbb{I} \to A$ so that $p\ i0 := a$ and $p\ i1 := b$.

- For types A and B, we have a function
  $$\text{transport} : (A \equiv B) \to (A \to B)$$

A path $p : a \equiv b$ is a function $p : I \to A$
so that $p\,0 := a$ and $p\,1 := b$.

For types A and B, we have a function
$\text{transport} : (A \equiv B) \to (A \to B)$

So if $x : A \vdash B(x) : \text{Type}$, and $p : a \equiv b$
then $\text{transport}\,(B \circ p) : B(a) \to B(b)$



The Main Theorem

Definition 2 (Homotopy data structure of conformal blocks)**.** In specialization of Def. 1, we obtain this data type:

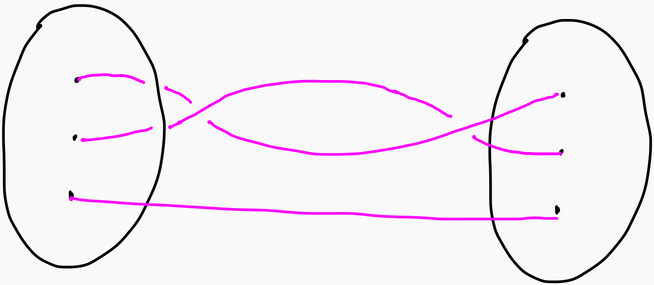
$$
\left.
\begin{array}{l}
\text{punctures} \quad \text{degree} \quad \text{shifted level} \\
N:\mathbb{N}_+, \quad n:\mathbb{N}, \quad \kappa:\mathbb{N}_{\geq 2} \\
\mathrm{w}_{(-)} : N \to \{0,\cdots,\kappa-2\} \\
\text{weights}
\end{array}
\right\}
\vdash
\left(\vec{z} \mapsto \overbrace{\left[(t : \mathbf{B}\mathbb{C}^{\times}) \to \left(\mathrm{fib}_{(t,\vec{z})}\left(\mathrm{pr}_N^{N+n},\, \tau_{(\kappa,\mathrm{w}_\bullet)}\right) \to \mathbf{B}^n\left(\circlearrowleft_t \mathbb{C}_{\mathrm{udl}} \right) \right) \right]_0}^{H^n_{\mathbb{C}}\left(\mathrm{Conf}(\mathbb{R}^2 \setminus \vec{z});\, \mathbb{C}\right)} \right) : \mathbf{B}\,\mathrm{PBr}(N) \to \mathrm{Type}
$$

$$
\begin{array}{cccc}
\mathrm{pr}_N^{N+n} : & \overbrace{\mathbf{B}\,\mathrm{PBr}(N+n)}^{\mathrm{Conf}_{N+n}(\mathbb{R}^2)} & \longrightarrow & \overbrace{\mathbf{B}\,\mathrm{PBr}(N)}^{\mathrm{Conf}_N(\mathbb{R}^2)} \\
& \mathrm{pt} \circlearrowleft b_{Ii} & \mapsto & \mathrm{pt} \circlearrowleft \mathrm{e} \\
& \mathrm{pt} \circlearrowleft b_{IJ} & \mapsto & \mathrm{pt} \circlearrowleft b_{IJ}
\end{array}
\qquad
\begin{array}{cccc}
\tau_{(\kappa,\mathrm{w}_\bullet)} : & \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\mathbb{C}^{\times} \\
& \mathrm{pt} \circlearrowleft b_{Ii} & \mapsto & \mathrm{pt} \circlearrowleft \exp(2\pi\mathrm{i}\,\frac{\mathrm{w}_I}{\kappa}) \\
& \mathrm{pt} \circlearrowleft b_{ij} & \mapsto & \mathrm{pt} \circlearrowleft \exp(2\pi\mathrm{i}\,\frac{2}{\kappa}) \\
& \mathrm{pt} \circlearrowleft b_{IJ} & \mapsto & \mathrm{pt} \circlearrowleft \exp(2\pi\mathrm{i}\,\frac{\mathrm{w}_I \mathrm{w}_J}{2\kappa})
\end{array}
\tag{3}
$$

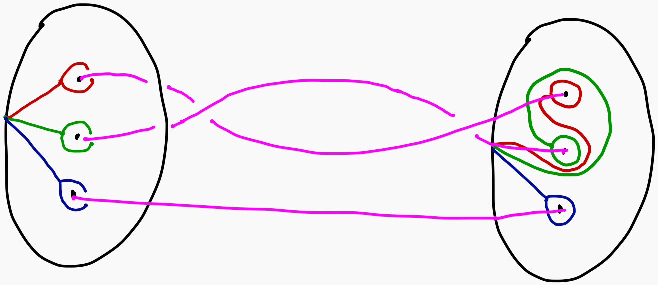
Theorem 3 (Topological quantum gates as homotopy data structure). *The semantics in the classical model topos of the transport operation (2) in this data type (3) is given by the monodromy of the Knizhnik-Zamolodchikov connection, on* $\widehat{\mathfrak{su}}_2{}^{\kappa-2}$*-conformal blocks (on the Riemann sphere with* $N+1$ *punctures weighted by* $(\mathrm{w}_I)_{I=1}^N$ *and* $\mathrm{w}_{N+1} = n + \sum_I \mathrm{w}_I$*).*

But how do we define the configuration spaces?

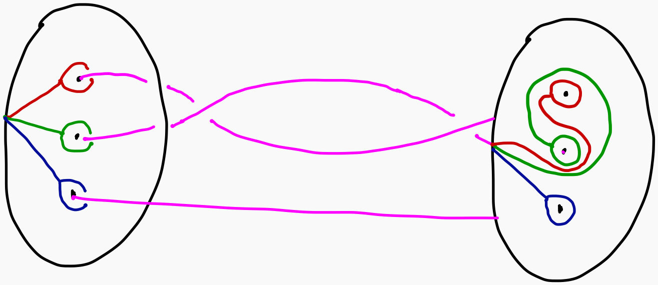
Braids as automorphisms of punctured disks



Braids as automorphisms of punctured disks

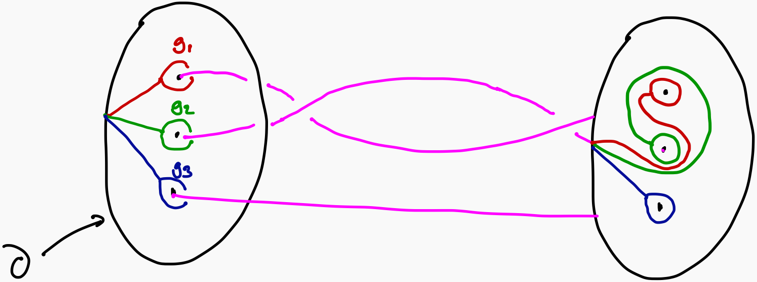


Braids as automorphisms of punctured disks



$$S^1 \vee S^1 \vee S^1 \xrightarrow{\sim} S^1 \vee S^1 \vee S^1$$

Braids as automorphisms of punctured disks



$$
\begin{tikzcd}
 & S^1\vee S^1\vee S^1 \arrow[rr, "\sim"] & & S^1\vee S^1\vee S^1 \\
S^1+S^1+S^1 \arrow[ur, "g_1g_2g_3"] \arrow[rr, equal] & & S^1+S^1+S^1 \arrow[ur] & \\
 & S^1 \arrow[uu, "\partial"'] \arrow[rr, equal] & & S^1 \arrow[uu, "\partial"']
\end{tikzcd}
$$

Braids as automorphisms of punctured disks

$$\text{"}\mathrm{Conf}_d(\mathbb{R}^2)\text{"} :\equiv \begin{cases} (C : \mathrm{Type}) \times \\ (g : d \times S^1 \to C) \times \\ (\partial : S^1 \to C) \times \\ \exists\,(\bigvee_d S^1 \simeq C) \end{cases}$$

$$\left(\bigvee_d S^1, g, \partial, \cdot\right) = \left(\bigvee_d S^1, g, \partial, \cdot\right) \simeq PBr(d)$$

$$\begin{tikzcd}
 & S^1 \vee S^1 \vee S^1 \arrow[rrr, "\sim"] & & & S^1 \vee S^1 \vee S^1 \\
S^1 + S^1 + S^1 \arrow[ur, "g_1\, g_2\, g_3"] \arrow[rrr, equal] & & & S^1 + S^1 + S^1 \arrow[ur] & \\
 & S^1 \arrow[uu, "\partial"'] \arrow[rrr, equal] & & & S^1 \arrow[uu, "\partial"']
\end{tikzcd}$$

Current and Future Work

- Mitchell Riley and I ran a research program for undergraduates at NYUAD, formalizing this approach in Cubical Agda.
  - ↳ Lecture notes and papers forthcoming!

- Linear homotopy type theory applied to TQC.

- Compositionality?