Modal Fracture of Higher Groups

$$
\begin{tikzcd}
 & B^k_\nabla \mathbb{R} \arrow[rr, "\text{curvature}"] \arrow[dr, "\text{Universal } \infty\text{-cover}"] & & \Lambda^{k+1}_{cl} \arrow[dr, "\text{de Rham class}"] & \\
\flat B^k \mathbb{R} \arrow[ur, "\text{Forget Flatness}"] \arrow[dr, "\exp"'] & & B^k_\nabla U(1) \arrow[ur, "\text{curvature}"] \arrow[dr, "\text{Top. Class}"] & & \flat B^{k+1}\mathbb{R} \\
 & \flat B^k U(1) \arrow[ur, "\text{Forget Flatness}"] \arrow[rr, "\beta"'] & & B^{k+1}\mathbb{Z} \arrow[ur, "\text{inclusion}"'] &
\end{tikzcd}
$$

arXiv: 2106.15390

David Jaz Myers

Johns Hopkins

at HoTT/UF 2021

A bit of history:

$$
(1) \quad
\begin{tikzcd}
 & \Lambda^k(X)/\mathrm{im}\, d \arrow[rr] \arrow[dr] & & \Lambda^{k+1}_{cl}(X) \arrow[dr] & \\
H^k(X;\flat\mathbb{R}) \arrow[ur] \arrow[dr] & & D_k(X) \arrow[ur] \arrow[dr] & & H^{k+1}(X;\flat\mathbb{R}) \\
 & H^k(X;\flat U(1)) \arrow[ur] \arrow[rr] & & H^{k+1}(X;\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

- In 1973, Cheeger and Simons introduced <u>differential characters</u>
- This is Deligne cohomology in the differential geometric setting

  "Ordinary differential cohomology"
- In 2008, Simons and Sullivan showed that the <u>character diagram</u> (1) characterizes ordinary diff. coh.

A bit more history:

- Bunke, Nikolaus, & Völkl (2012) construct diff. coh. theories as sheaves of spectra on smooth manifolds

$$\begin{tikzcd}
 & \mathcal{A}(\hat{E})^k(X) \arrow[rr] \arrow[dr] & & \mathcal{Z}(\hat{E})^k(X) \arrow[dr] & \\
H^{k-1}(X; Z(\hat{E})) \arrow[ur] \arrow[dr] & & \hat{E}^k(X) \arrow[ur] \arrow[dr] & & H^k(X; Z(\hat{E})) \\
 & H^k(X; S(\hat{E})) \arrow[rr] \arrow[ur] & & H^k(X; U(E)) \arrow[ur] &
\end{tikzcd}$$

- Schreiber (2013) shows how these hexagons arise from any adjoint <u>modality</u>/<u>comodality</u>

$$\begin{tikzcd}
 & \Pi_{\mathrm{dR}}\hat{E} \arrow[rr, "\mathbf{d}"] \arrow[dr] & & \flat_{\mathrm{dR}}\hat{E} \arrow[dr] & \\
\Pi_{\mathrm{dR}}\flat\hat{E} \arrow[ur] \arrow[dr] & & \hat{E} \arrow[ur] \arrow[dr] & & \Pi\flat_{\mathrm{dR}}\hat{E} \\
 & \flat\hat{E} \arrow[rr] \arrow[ur] & & \Pi\hat{E} \arrow[ur] &
\end{tikzcd}$$

Modal Fracture of Higher Groups:

Thm: For any crisp higher group G, (Unstable version of Schreiber §4.1.2)

$$\begin{tikzcd}
 & \overset{\infty}{\hat{G}} \arrow[rr, "\theta"] \arrow[dr, "\pi"] & & \mathfrak{g} \arrow[dr, "(-)^{\int}"] & \\
\flat\overset{\infty}{\hat{G}} \arrow[ur, "(-)_\flat"] \arrow[dr, "\flat\pi"'] & & G \arrow[ur, "\theta"] \arrow[dr, "(-)^{\int}"] & & \int\mathfrak{g} = \flat B\overset{\infty}{\hat{G}} \\
 & \flat G \arrow[rr, "\beta"'] \arrow[ur, "(-)_\flat"] & & \int G \arrow[ur, "\int\theta"'] &
\end{tikzcd}$$

Where
- $\overset{\infty}{\hat{G}} \xrightarrow{\pi} G$ is the <u>universal ∞-cover</u>
- $G \xrightarrow{\theta} \mathfrak{g}$ is the <u>infinitesimal</u> <u>remainder</u>

And
- Both squares are pullbacks.
- The top and bottom sequences are fiber sequences.
- $\int\mathfrak{g} = \flat B\overset{\infty}{\hat{G}}$

Cohesive HoTT - Crispness and $\flat$-comodality (Shulman)

$$\Delta \mid \Gamma \vdash a : A$$

Add crisp variables to express discontinuous dependence

$$x :: A$$

Crisp terms: $\Delta \mid \cdot \vdash a : A$ have only crisp variables.

Comodality $\flat$: $\flat A$ is inductively generated by crisp $a :: A$.

$$\frac{\Delta \mid \cdot \vdash A : \mathrm{Type}}{\Delta \mid \Gamma \vdash \flat A : \mathrm{Type}} \qquad \frac{\Delta \mid \cdot \vdash a : A}{\Delta \mid \Gamma \vdash a^\flat : \flat A}$$

$$\frac{\begin{array}{c}\Delta \mid \Gamma, x : \flat A \vdash C : \mathrm{Type} \\ \Delta \mid \Gamma \vdash a : \flat A \\ \Delta, x :: A \mid \Gamma \vdash c : C(x^\flat)\end{array}}{\Delta \mid \Gamma \vdash \mathrm{let}\ x^\flat :\equiv a \ \mathrm{in}\ c : C(a)}$$

$$(\mathrm{let}\ x^\flat :\equiv a^\flat \ \mathrm{in}\ c \equiv c(a))$$

Counit: $(-)_\flat : \flat A \to A$

$$a^\flat \mapsto a$$

$$u \mapsto \mathrm{let}\ a^\flat :\equiv u \ \mathrm{in}\ a.$$

The shape modality $\int$

The "shape" modality $\int$ reflects into discrete types.

- In differential geometry, $\int \equiv \mathrm{Loc}_{\mathbb{R}}$ takes the homotopy type.
- In simplicial cohesion, $\int \equiv \mathrm{Loc}_{\Delta^1}$ is the geometric realization.

e.g. $S^n = \{x : \mathbb{R}^{n+1} \mid |x| = 1\}$, then $\int S^n = S^n$ higher inductive.

Axiom: For a crisp type X

$$\flat X \xrightarrow[(-)_\flat]{\sim} X \quad \text{iff} \quad X \xrightarrow[(-)^\int]{\sim} \int X$$

In either case, we say X is <u>discrete</u>

Thm (Shulman): For crisp X and Y

$$\flat\left(\int X \to Y\right) \simeq \flat(X \to \flat X)$$

The universal ∞-cover and infinitesimal remainder:

Def: A higher group is a type G equipped with a pointed, 0-connected type BG (called the "delooping") with

$$G = \Omega BG$$

Def: Let G be a crisp higher group

- The <u>universal</u> <u>∞-cover</u> is the fiber

$$\begin{tikzcd} \overset{\infty}{G} \arrow[r, "\pi"] & G \arrow[r, "(-)^{\int}"] & \int G \arrow[dll, out=0, in=180] \\ B\overset{\infty}{G} \arrow[r] & BG \arrow[r] & \int BG \end{tikzcd}$$

- The <u>infinitesimal</u> <u>remainder</u> is the fiber

$$\begin{tikzcd} \flat G \arrow[r, "(-)_\flat"] & G \arrow[r, "\theta"] & \mathfrak{g} \arrow[dll, out=0, in=180] \\ \flat BG \arrow[r] & BG \end{tikzcd}$$

The Pullback Squares

Prop: For G a crisp higher group,

$$\begin{tikzcd} & \overset{\infty}{G} \arrow[dr, "\pi"] & \\ \flat\overset{\infty}{G} \arrow[ur, "(-)_\flat"] \arrow[dr, "\flat\pi"'] \arrow[rr, phantom, ">"] & & G \\ & \flat G \arrow[ur, "(-)_\flat"'] & \end{tikzcd}$$

is a pullback.

Proof: The fiber $\mathrm{fib}_\pi(g_\flat)$ is identifiable with $\Omega \int G$, which is crisply discrete. □

Prop: For G a crisp higher group,

$$\begin{tikzcd} & \mathfrak{g} \arrow[dr, "(-)^{\int}"] & \\ G \arrow[ur, "\theta"] \arrow[dr, "(-)^{\int}"'] \arrow[rr, phantom, ">"] & & \int \mathfrak{g} \\ & \int G \arrow[ur, "\int\theta"'] & \end{tikzcd}$$

is a pullback.

Proof: The fibers of θ are identifiable with $\flat G$, which is crisply discrete.

By the "good fibrations" trick, (Thm 6.1 of "Good fibrations...") this implies that θ is an ∞-cover. □

The Modal Fracture Hexagon

$$
\begin{tikzcd}
 & \overset{\infty}{G} \arrow[rr] \arrow[dr] & & \mathfrak{g} \arrow[dr] & \\
b\overset{\infty}{G} \arrow[ur] \arrow[dr] & & G \arrow[ur] \arrow[dr] & & \int\mathfrak{g} \\
 & bG \arrow[ur] \arrow[rr] & & \int G \arrow[ur] &
\end{tikzcd}
$$

universal ∞-cover

infinitesimal remainder

$$\overset{\infty}{G} \to G \to \int G \to B\overset{\infty}{G} \to BG \to \int BG$$

$$bG \to G \to \mathfrak{g} \to bBG \to BG$$

$$b \big\downarrow \qquad\qquad\qquad\qquad \big\downarrow \int$$

Using the theory of $\int$-fibrations

$$b\overset{\infty}{G} \to bG \to \int G \to bB\overset{\infty}{G} \to bBG \to \int BG \quad = \quad bG \to \int G \to \int\mathfrak{g} \to bBG \to \int BG$$

So, $\int\mathfrak{g} = bB\overset{\infty}{G}$

Ordinary Differential Cohomology

Assumption: We have "form classifiers" Λ^k, abelian groups and an exact sequence

$$0 \to b\mathbb{R} \to \mathbb{R} \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \to \cdots$$

Where $\int\Lambda^k = *$

Define $\Lambda^k_{cl} :\equiv \mathrm{Ker}(\Lambda^k \xrightarrow{d} \Lambda^{k+1})$, and assume $b\Lambda^k_{cl} = *$

Thm: $\int\Lambda^k_{cl} = bB^k\mathbb{R}$ (the de Rham class)

Def: The classifiers for circle k-gerbes with connection is the pullback:

$$
\begin{tikzcd}
B^k_\nabla U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda^{k+1}_{cl} \arrow[d, "(-)^\int"] \\
B^{k+1}\mathbb{Z} \arrow[r] & bB^{k+1}\mathbb{R}
\end{tikzcd}
$$

Thm: The modal fracture hexagon of $B^k_\nabla U(1)$ is

$$
\begin{tikzcd}
 & B^k_\nabla \mathbb{R} \arrow[rr, "\text{curvature}"] \arrow[dr, "\text{Universal } \infty\text{-cover}"] & & \Lambda^{k+1}_{cl} \arrow[dr, "\text{de Rham class}"] & \\
bB^k\mathbb{R} \arrow[ur, "\text{Forget Flatness}"] \arrow[dr, "\exp"'] & & B^k_\nabla U(1) \arrow[ur, "\text{Curvature}"] \arrow[dr, "\text{Top. class}"] & & bB^{k+1}\mathbb{R} \\
 & bB^k U(1) \arrow[rr, "\beta"'] \arrow[ur, "\text{Forget Flatness}"] & & B^{k+1}\mathbb{Z} \arrow[ur, "\text{inclusion}"'] &
\end{tikzcd}
$$

References: Thank You!

David Jaz Myers:
- Modal Fracture of Higher Graps (arXiv: 2106.15309)
- Good fibrations through the modal prism (arXiv: 1908.08034)

Urs Schreiber:
- Differential Cohomology in a cohesive ∞-topos (arXiv: 1310.7390)
- Differential Cohesion and Idelic Structure (nLab)

Mike Shulman:
- Brouwer's Fixed Point Theorem in Real Cohesive HoTT (1509.07584)
- & Bas Spitters – Modalities in HoTT (arXiv: 1706.07526)

Egbert Rijke
- Classifying Types (arXiv: 1906.09435)
- Modal Descent (arXiv: 2003.09713)

Felix Cherubini:
- Cohesive Covering Theory

James Simons and Dennis Sullivan:
- Axiomatic Characterization of Ordinary Differential Cohomology (arXiv: math/0701077)