Higher Groups in Homotopy Type Theory

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Part 1: Homotopy Type Theory

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- To identify $\mathsf{H}^n(\mathbb{S}^n; \mathbb{Z})$ with $\mathbb{Z}$, we must choose an orientation for the n-sphere $\mathbb{S}^n$.
- To identify the natural number n such that $\pi_4(\mathbb{S}^3)$ is isomorphic to $\mathbb{Z}_{/n}$ with the number 2, we need to prove that n equals 2.

Type Theory

A *type* is a type of mathematical thing.

Type theory gives *rules* for making new types and new terms of them.
It is a full foundation of mathematics, from scratch.

$$\text{a term} : \text{its type}$$

$$a : A$$

$$3 : \mathbb{N}$$

$$\mathbb{N} : \mathbf{Set}$$

$$T_pM : \mathbf{Vect}_{\mathbb{R}}$$

$$\mathbf{Vect}_{\mathbb{R}} : \mathbf{Type}$$

$$K(\mathbb{Z}; 2) : \mathbf{Type}$$

Identifications

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Axiom (Univalence)

If X and Y are types, then $X = Y$ is the type of equivalences of X with Y.

Dependent Types and Functions

Types can depend on elements of others types.

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- A vector field is $v : (p : M) \to T_pM$.

Dependent Types and Pairs

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is the type of pairs (a, b) with $a : A$ and $b : B(a)$.

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Definition

For a function $f : A \to B$ and $b : B$, its *fiber* is the type

$$\mathrm{fib}_f(b) :\equiv (a : A) \times (f(a) = b)$$

together with the map $(a, p) \mapsto a : \mathrm{fib}_f(b) \to A$.

Contractible Types and the Substitution Lemma

Definition

A *center of contraction* for a type A is an element $c : A$ such that for every other element $a : A$, we have an identification of a with c.

$$\mathbf{Contr}(A) :\equiv (c : A) \times \big((a : A) \to (a = c)\big)$$

Definition

A map $f : A \to B$ is an *equivalence* if its fibers are contractible:

$$\mathbf{Equiv}(f) :\equiv (y : Y) \to \mathbf{Contr}(\mathrm{fib}_f(y))$$

Contractible Types and the Substitution Lemma

Lemma (Singleton Lemma, UFP)

For any type A and element $c : A$, the singleton type ("based path type")

$$(a : A) \times (a = c)$$

is contractible.

Lemma (Substitution Lemma, UFP)

Suppose $B : A \to \textbf{Type}$ and $c : \textbf{Contr}(A)$ is a center of contraction for A. Then

$$(a : A) \times B(a) = B(c).$$

The Long Fiber Sequence from a Map

Corollary (Substitution Lemma)

Let $c : A$ and suppose that $B : A \to \textbf{Type}$. Then

$$(a : A) \times (a = c) \times B(a) = B(c).$$

The Long Fiber Sequence from a Map

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$$\mathrm{fib}_f(y_0) \xrightarrow[\mathrm{fst}]{} X \xrightarrow[f]{} Y$$

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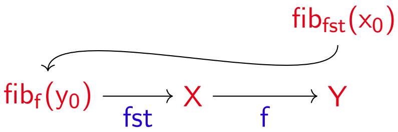

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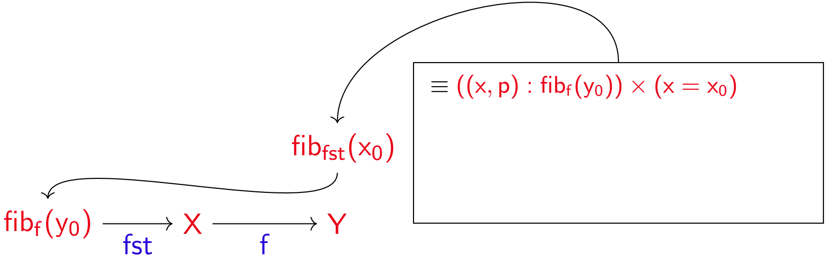

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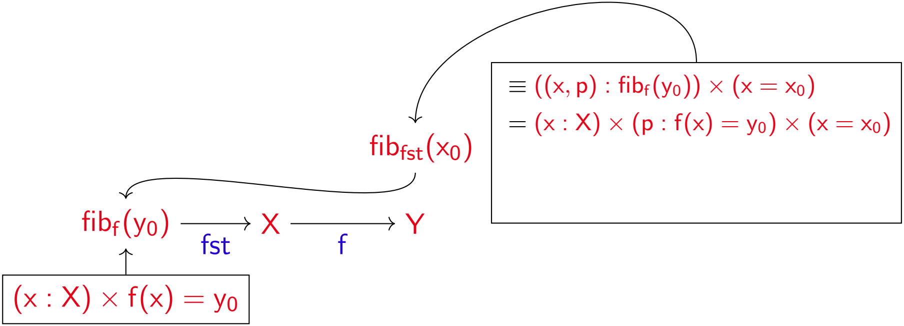

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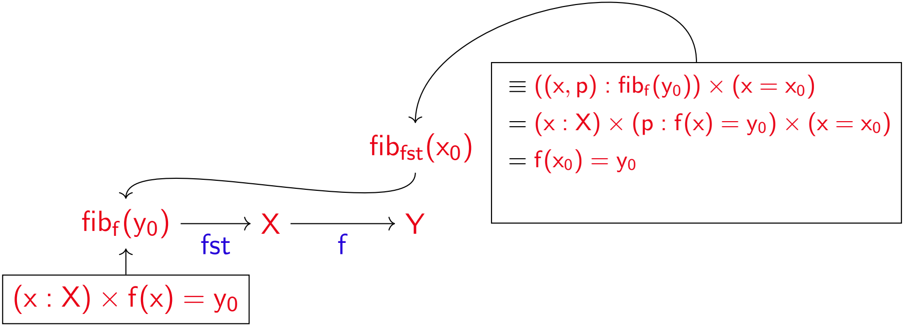

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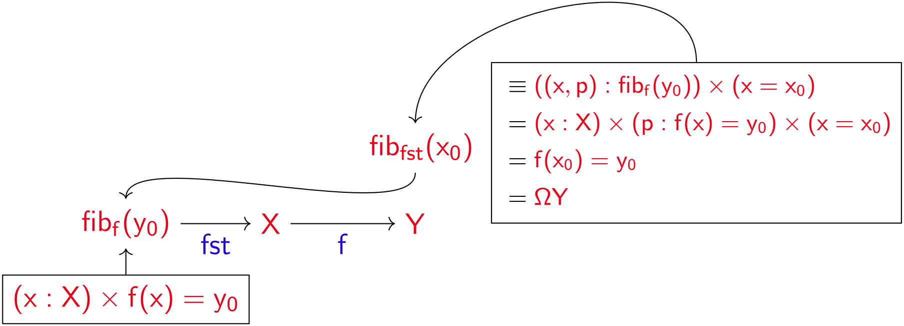

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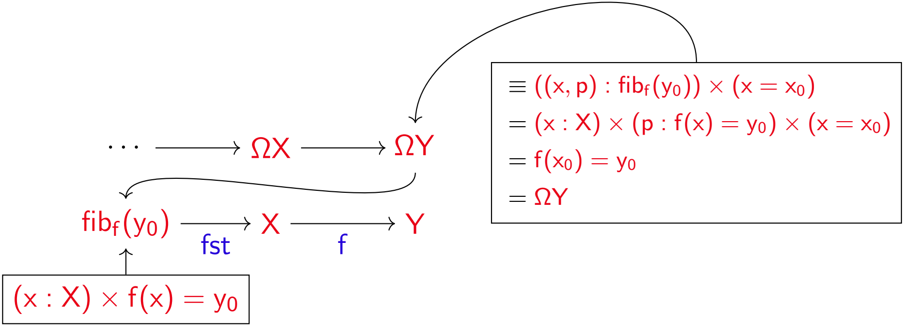

Propositions, Sets, and More

Lemma (UFP)

For any two centers of contraction $c, d : \mathbf{Contr}(A)$, $c = d$ is contractible.

Definition

- A *proposition* is a type P such that for any $a, b : P$, $a = b$ is contractible.

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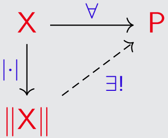
- A *groupoid* is a type G such that for any $a, b : G$, $a = b$ is a set.
- ...
- An n-*type* is a type X such that for any $a, b : X$, $a = b$ is an $(n-1)$-type (with -2-types being contractible).

Truncation

Theorem (UFP)

For any type X, there is a proposition $\|X\|$ and a map $|\cdot| : X \to \|X\|$ and such that



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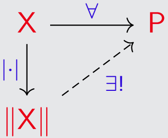

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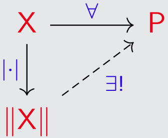

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- $\|(a : A) \times B(a)\|$ represents the proposition $\exists a : A.\, B(a)$.
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Part 2: Higher Groups

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- The dihedral group D_{2n} is the type of symmetries of an n-gon in the plane.
- ...

- Let $\mathsf{BAut}(\mathsf{n}) :\equiv (\mathsf{X} : \mathbf{Set}) \times \|\mathsf{X} = \{1, \ldots, \mathsf{n}\}\|$ be the type of n-element sets, pointed at $\{1, \ldots, \mathsf{n}\}$. Then

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Definition

For $a : A$, define

$$\mathsf{BAut}_A(a) :\equiv (b : A) \times \|b = a\|$$

be the type of things *identifiable* with a, pointed at a. Then

$$\mathsf{Aut}_A(a) \equiv a =_A a = \Omega\,\mathsf{BAut}_A(a)$$

∞-Groups

Definition

An ∞-*group* is a type G identified with the loops of a pointed, connected type BG (the *delooping* of G).

$$\begin{aligned}\infty\mathsf{Group} :\equiv (G : \mathbf{Type}) \times (BG : \mathbf{Type}_*^{>0}) \times G = \Omega\, BG \\ = \mathbf{Type}_*^{>0}.\end{aligned}$$

Definition

An abstract group is a set G equipped with associative, unital, and invertible binary operation.

Groups are ∞-Groups

Definition

For any abstract group G, the type

$$BG :\equiv \{G\text{-torsors}\}$$

is connected, and pointed at G has

$$\Omega\, BG = G.$$

Definition

A *homomorphism* of ∞-groups is a pointed map between their deloopings:

$$\mathrm{BG} \cdot\!\!\to \mathrm{BH} :\equiv (\mathrm{f} : \mathrm{BG} \to \mathrm{BH}) \times (\mathrm{f}(\mathrm{pt}_{\mathrm{BG}}) = \mathrm{pt}_{\mathrm{BH}})$$

- Consider $\mathrm{f} \equiv \mathrm{X} \mapsto \mathrm{X} + 1 : \mathrm{BAut}(\mathrm{n}) \cdot\!\!\to \mathrm{BAut}(\mathrm{n}+1)$,
  - pointed by $\mathrm{e} : \{1, \ldots, \mathrm{n}\} + \{\mathrm{n}+1\} = \{1, \ldots, \mathrm{n}+1\}$.

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- Consider $\mathrm{f} \equiv \mathrm{V} \mapsto \Lambda^{\mathrm{n}}\mathrm{V} : \mathrm{B}\mathbf{GL}_{\mathrm{n}}(\mathbb{R}) \cdot\!\!\to \mathrm{B}\mathbf{GL}_1(\mathbb{R})$,
  - pointed by $\mathrm{e}_1 \wedge \cdots \wedge \mathrm{e}_{\mathrm{n}} \mapsto 1 : \Lambda^{\mathrm{n}}\,\mathbb{R}^{\mathrm{n}} = \mathbb{R}$.

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- Consider $f \equiv V \mapsto \Lambda^n V : \mathsf{B}\mathbf{GL}_n(\mathbb{R}) \cdot\!\!\to \mathsf{B}\mathbf{GL}_1(\mathbb{R})$,
  - pointed by $e_1 \wedge \cdots \wedge e_n \mapsto 1 : \Lambda^n \mathbb{R}^n = \mathbb{R}$.
  - Then $\Omega f = \det$ takes the determinant.

Schreier Theory: Classifying Extensions

Lemma

For groups F and G the type of extensions $F \to E \to G$ of G by F is equivalent to the type of fiber sequences $BF \cdot\!\!\to BE \cdot\!\!\to BG$.

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Theorem (Schreier Theory for ∞-groups)

Let F and G be ∞-groups, then

$$\{\text{Extensions of } G \text{ by } F\} = BG \cdot\!\!\to \text{BAut}(BF)$$

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$$\{\text{Extensions of } \mathsf{G} \text{ by } \mathsf{F}\} = \mathsf{BG} \cdot\!\!\rightarrow \mathsf{BAut}(\mathsf{BF})$$

Corollary

For any group G,

$$\{\text{Extensions of } \mathbb{Z} \text{ by } \mathsf{G}\}_{/\cong} = \mathbf{Out}(\mathsf{G})$$

References

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