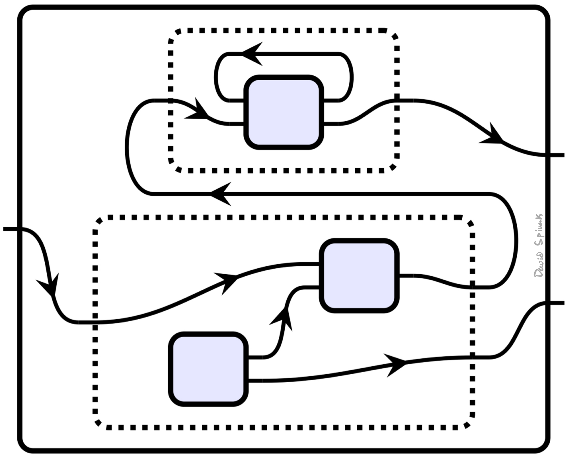
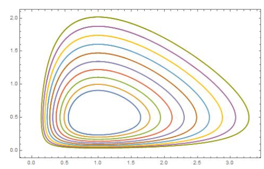
Doctrines of Dynamical Systems



David Jaz Myers

Center for Quantum & Topological Systems at NYU AD

Logic		Systems Theory	
6	Element	Behavior	
$\mathbb{Z}/7$	Model	System	$\begin{cases} \frac{dR}{dt} = aR - FR \\ \frac{dF}{dt} = RF - dF \end{cases}$
Group	Theory	System Theory	Systems of ODEs
Algebraic Theory	Doctrine	Doctrine	Parameter-Setting (Lenses)

What is a System? "a whole composed of interacting parts"

- Discrete time dynamical system
- Markov decision process
- (Non)Deterministic automaton/Moore Machine
- System of ODEs
- Open game

- Hamiltonian/Port-Hamiltonian graph
- Lagrangian
- Willems-style sheaves of behaviors

- Petri Nets
- Circuits
- Networks and flow graphs
- Labelled transition systems
- Stock-flow models
- Quantum Circuits

...

System Theories

"What does it mean to be a 'system'"

Categorical Systems Theory

Main ideas:

Modular

- A system interacts with its environment through an **interface**.
- Complex systems are formed by component subsystems interacting through their interfaces via a **composition pattern**.
- Systems may **simulate** or **map onto** other systems.
- System mappings may also compose along composition patterns, so long as they agree on the interfaces.

A **compositionality theorem** expresses a way that behaviors, facts, or properties of composite systems may be deduced from their components and the composition pattern.

Categorical Systems theory

- Discrete time dynamical system
- Markov (decision) process
- (Non)Deterministic automaton/Moore Machine
- System of ODEs
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ALGEBRAS OF OPEN DYNAMICAL SYSTEMS ON THE OPERAD OF WIRING DIAGRAMS
DMITRY VAGNER, DAVID I. SPIVAK, AND EUGENE LERMAN

An Algebra of Resource Sharing Machines
Unifying Two Flavors of Open Dynamical Systems
Sophie Libkind

Coarse-Graining Open Markov Processes
John C. Baez, Kenny Courser

A Compositional Framework for Markov Processes
John C. Baez, Brendan Fong, Blake S. Pollard

Polynomial Functors: A General Theory of Interaction
Nelson Niu David I. Spivak

A Categorical Theory of Hybrid Systems
by
Aaron David Ames

A categorical approach to open and interconnected dynamical systems
Brendan Fong, Paolo Rapisarda, Paweł Sobociński

Compositional Game Theory
Neil Ghani Jules Hedges Viktor Winschel

Towards Foundations of Categorical Cybernetics
Matteo Capucci, Bruno Gavranović, Jules Hedges, Eigil Fjeldgren Rischel

OPEN SYSTEMS IN CLASSICAL MECHANICS
JOHN C. BAEZ[1], DAVID WEISBART[2], AND ADAM M. YASSINE[3]

DYNAMICAL SYSTEMS AND SHEAVES
PATRICK SCHULTZ, DAVID I. SPIVAK, AND CHRISTINA VASILAKOPOULOU

Open Petri Nets
John C. Baez, Jade Master

Structured Cospans
John C. Baez, Kenny Courser

Hypergraph Categories
Brendan Fong and David I. Spivak

A Compositional Framework for Passive Linear Networks
John C. Baez, Brendan Fong

Span(Graph): a Canonical Feedback Algebra of Open Transition Systems
Elena Di Lavore, Alessandro Gianola, Mario Román, Nicoletta Sabadini, Paweł Sobociński

Compositional Modeling with Stock and Flow Diagrams
John Baez, Xiaoyan Li, Sophie Libkind, Nathaniel Osgood, and Evan Patterson

CATEGORIES OF QUANTUM AND CLASSICAL CHANNELS
BOB COECKE, CHRIS HEUNEN, AND ALEKS KISSINGER

Topological Quantum Computation Through the Lens of Categorical Quantum Mechanics
Fatimah Rita Ahmadi and Aleks Kissinger

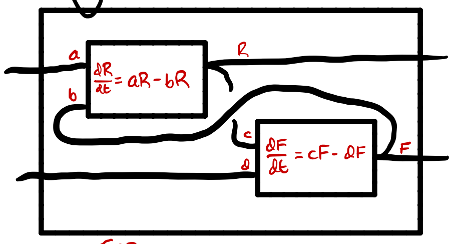
Doctrines of Systems

Informal definition:
A **Doctrine** is a suite of answers to the following questions:

1. What does it mean to be a system? (More Qs per theory)
2. What should the interface of a system be?
3. How can interfaces be connected in composition patterns?
4. How are systems composed through these composition patterns?
5. What is a map between systems?
6. When can maps be composed along the composition patterns?

3 General Ways to Compose

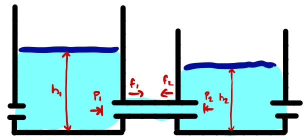
Setting parameters (Lenses)



$$\begin{cases} \frac{dR}{dt} = aR - FR \\ \frac{dF}{dt} = RF - \ell F \end{cases}$$

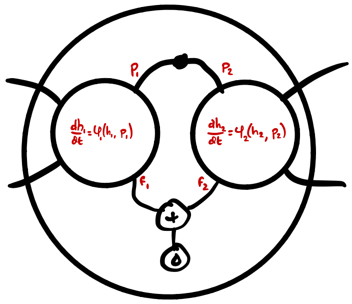
Interface: exposed variables

Sharing Variables (Spans)



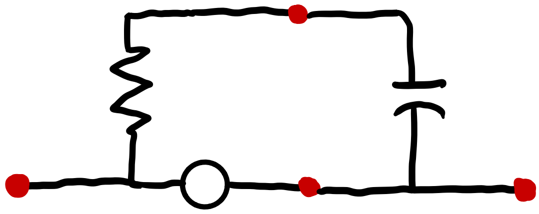
$P_1 = P_2$

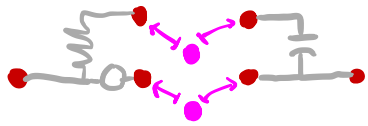
$0 = f_1 + f_2$



Interface: exposed variables

Plugging in Ports (Cospans)





Interface: exposed ports

- Discrete time dynamical system
- Markov decision process
- (Non)Deterministic automaton/Moore Machine
- System of ODEs
- Open games

} Parameters are set by variables of state (uses Lenses)

- Hamiltonian / Port-Hamiltonian graph
- Lagrangian
- Willems-style sheaves of behaviors

} Variables are shared between systems (uses Spans)

- Petri Nets
- Circuits
- Networks and flow graphs
- Labelled transition systems
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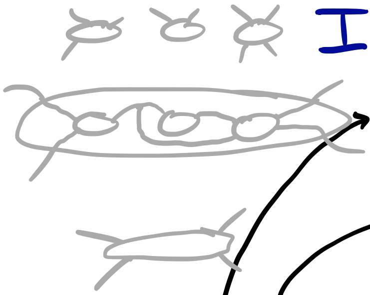
} Exposed ports are plugged into eachother (uses Cospans)

Process Theories as Monoidal Double Categories



$$\begin{tikzcd}
I_1 \otimes \cdots \otimes I_n \arrow[r, "f_1 \otimes \cdots \otimes f_n"] \arrow[d, "P"'] \arrow[dr, phantom, "\Rightarrow"] & J_1 \otimes \cdots \otimes J_n \arrow[d, "Q"] \\
I \arrow[r, "f"'] & J
\end{tikzcd}$$

Squares as maps of processes

Objects as interfaces

horizontals as maps

Vertical morphisms as processes

Useful idea: A composition pattern is a free process.

3 General Ways to Compose

Setting parameters (Lenses)

$$\begin{tikzcd}
\binom{I_1^-}{I_1^+} \otimes \cdots \otimes \binom{I_n^-}{I_n^+} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift right] & \binom{J_1^-}{J_1^+} \otimes \cdots \otimes \binom{J_n^-}{J_n^+} \arrow[d, shift right] \\
\binom{I^-}{I^+} \arrow[u, shift right] \arrow[r, shift left] \arrow[r, shift right] & \binom{J^-}{J^+} \arrow[u, shift right]
\end{tikzcd}$$

Interface: exposed variables

Sharing Variables (Spans)

$$\begin{tikzcd}
I_1 \times \cdots \times I_n \arrow[r] & J_1 \times \cdots \times J_n \\
P \arrow[u] \arrow[d] \arrow[r] & Q \arrow[u] \arrow[d] \\
I \arrow[r] & J
\end{tikzcd}$$

Interface: exposed variables

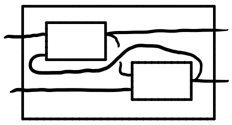
Plugging in Ports (Cospans)

$$\begin{tikzcd}
I_1 \times \cdots \times I_n \arrow[r] \arrow[d] & J_1 \times \cdots \times J_n \arrow[d] \\
P \arrow[r] & Q \\
I \arrow[u] \arrow[r] & J \arrow[u]
\end{tikzcd}$$

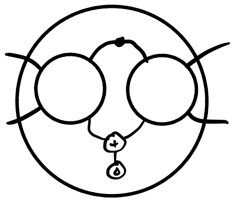
Interface: exposed ports

Wiring Diagrams are Free Processes

- Lenses make sense in any cartesian category.
  - The free cartesian category is FinSet^{op}.
  - Lenses in FinSet^{op} are wiring diagrams:



- Spans make sense in any finitely complete category.
  - The free fin. complete cat is FinSet^{op}
  - Spans in FinSet^{op} are bubble diagrams.



- Cospans make sense in any finitely cocomplete category
  - The free fin. cocomplete cat is FinSet.
  - Cospans in FinSet are bubble diagrams.

Systems Theories as Monoidal Double Copresheaves

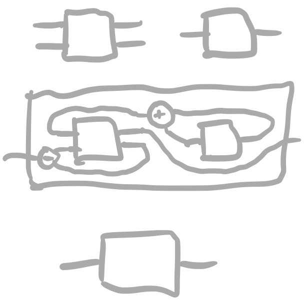
If $\mathbb{P}$ is a process theory (monoidal double cat),

Then a systems theory composed via $\mathbb{P}$ is

a lax monoidal lax double functor

$$\mathsf{Sys} : \mathbb{P}^T \longrightarrow \mathbf{Span}(\mathsf{Set}).$$

large sets



$I_1 \otimes I_2 \longmapsto \{\text{systems with interface } I_1\} \times \{\cdots I_2\}$

$P \longmapsto$ compose along P

$J \longmapsto \{\text{systems with interface } J\}$

Doctrines

Def: A *Doctrine* is a 2-functor

$$\mathrm{Sys}^{\mathbb{D}} : \mathrm{Theory}(\mathbb{D}) \longrightarrow \mathrm{MonDblCoPsh}$$

Three Examples:

- Parameter-Setting: $\left\{\begin{tikzcd} & \text{Bundles} \arrow[d] \\ T \arrow[ur, dashed] \arrow[r, dashed] & \text{Spaces} \end{tikzcd}\right\} \longrightarrow \mathrm{MonDblCoPsh}$
- Variable-Sharing: $\mathrm{FinitelyCompleteCat} \longrightarrow \mathrm{MonDblCoPsh}$
- Port-Plugging: $\mathrm{FinitelyCoCompleteCat} \longrightarrow \mathrm{MonDblCoPsh}$
  -or-
  $\mathrm{StructuredCospans} \longrightarrow \mathrm{MonDblCoPsh}$

Compositionality Theorems as Maps of Systems Theories

Generally mapping into the variable-sharing doctrine.

Thm: Representable maps of systems theories

Given $I \in \mathbb{P}$ and $S \in \mathrm{Sys}(I)$, get

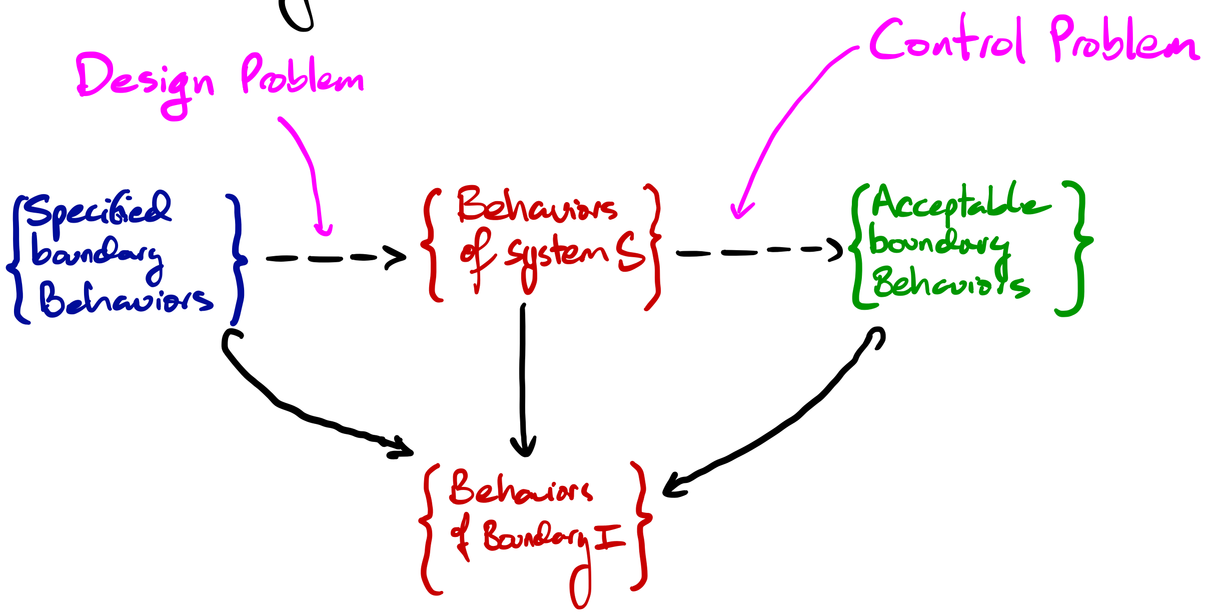
$$\begin{tikzcd} \mathbb{P} \arrow[rr, bend left, "\mathrm{Sys}"] \arrow[d, dashed, "\mathbb{P}(I,-)"'] & \arrow[d, phantom, "\Downarrow \mathrm{Sys}(S,-)"] & \mathrm{Span}(\mathrm{Set}) \\ \mathrm{Span}(\mathrm{Set})^{T} \arrow[urr, "\mathrm{Set}/(-)"'] & & \end{tikzcd}$$

$$\begin{tikzcd} T \in \mathrm{Sys}(J) \arrow[d, mapsto, dashed] \\ \left\{\begin{matrix}\text{Maps}\\ S \to T\end{matrix}\right\} \arrow[d] \\ \left\{\begin{matrix}\text{Maps}\\ I \to J\end{matrix}\right\} \end{tikzcd}$$

Behaviors of shape S in T.

Behaviors of the interface.

Control & Design Problems



In some cases, universal solutions to control & design problems exist by the adjoint functor theorem.

Draft book @ DavidJaz.com

Thank You!