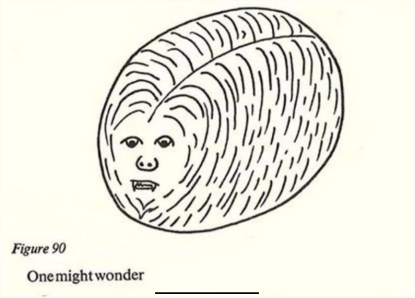
Tangent Bundles of Spheres



David Jaz Myers
jww: Ulrik Buchholtz
Dan Christensen
Egbert Rijke

- This is a synthetic homotopy theory talk.
  - Homotopy theory, but just w/ universal properties.

- We will use Book HoTT throughout.
  - pair types $(a:A)\times B(a)$ "Σ-types"
  - function types $(a:A)\to B(a)$ "Π-types"
  - natural numbers $\mathbb{N}$ and induction
  - identity types $(a=b)$ and path induction
  - Univalent universes $\mathcal{Type}$
  - pushouts
    $gl : (a:A)\to(\mathrm{inl}(fa)=\mathrm{inr}(g(a)))$

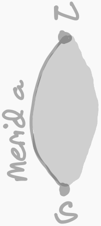
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{gl}{\Nearrow}"] & B \arrow[d, "\mathrm{inr}"] \\ C \arrow[r, "\mathrm{inl}"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

! - Banded types and Cohomology.

- **Pushouts** specialize to a number of important constructions

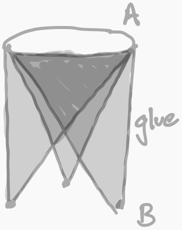
- **Suspension**

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\mathrm{Merid}}{\Nearrow}"] & * \arrow[d, "S"] \\ * \arrow[r, "N"'] & \Sigma A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$



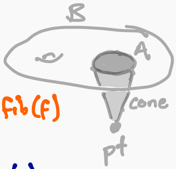
- **Join**

$$\begin{tikzcd} A \times B \arrow[r, "\mathrm{snd}"] \arrow[d, "\mathrm{fst}"'] \arrow[dr, phantom, "\overset{\mathrm{glue}}{\Nearrow}"] & B \arrow[d, "\mathrm{inr}"] \\ A \arrow[r, "\mathrm{inl}"'] & A * B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$



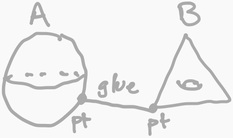
- **Cofiber**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "!"'] \arrow[dr, phantom, "\overset{\mathrm{cone}}{\Nearrow}"] & B \arrow[d, "\mathrm{cofib}(f)"] \\ * \arrow[r, "\mathrm{pt}"'] & \mathrm{Cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$



- **Wedge**

$$\begin{tikzcd} * \arrow[r, "\mathrm{pt}"] \arrow[d, "\mathrm{pt}"'] \arrow[dr, phantom, "\overset{\mathrm{glue}}{\Nearrow}"] & B \arrow[d] \\ A \arrow[r] & A \vee B \end{tikzcd}$$



- Spheres and Cubes are iterated pushouts

Spheres

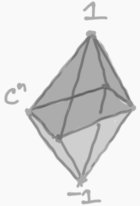
$S^{-1} :\equiv \emptyset$

$S^{n+1} :\equiv \Sigma S^n$

Cubes

$C^0 :\equiv \{-1, 1\}$

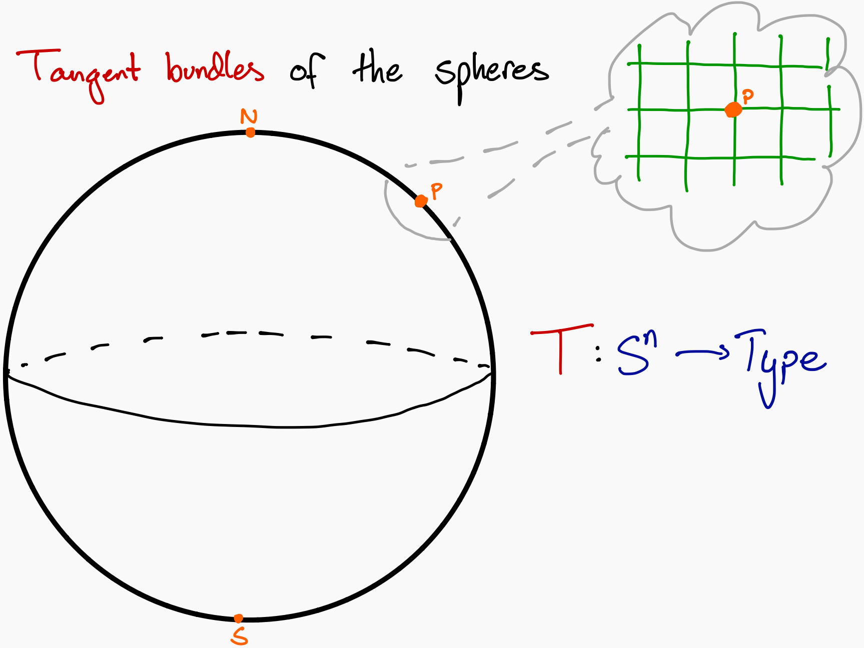
$C^{n+1} :\equiv C^0 * C^n$



- Lemma (Book):
  For any type A, $\Sigma A \simeq C^0 * A$

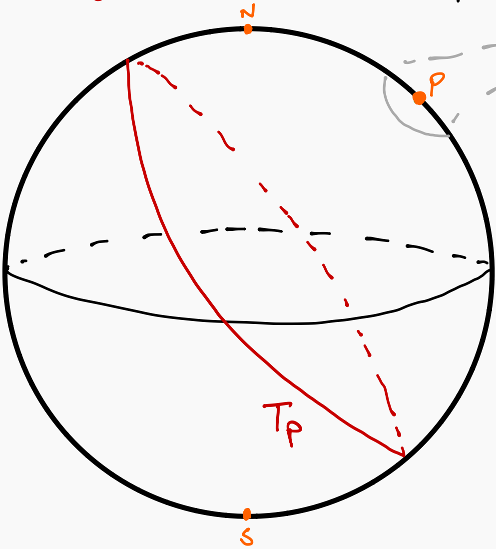
- Corollary: $S^n \simeq C^n$

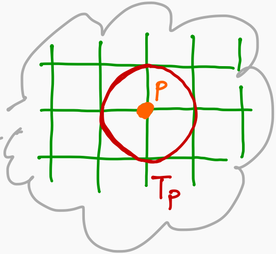
- Tangent bundles of the spheres



$$T : S^n \to \mathrm{Type}$$

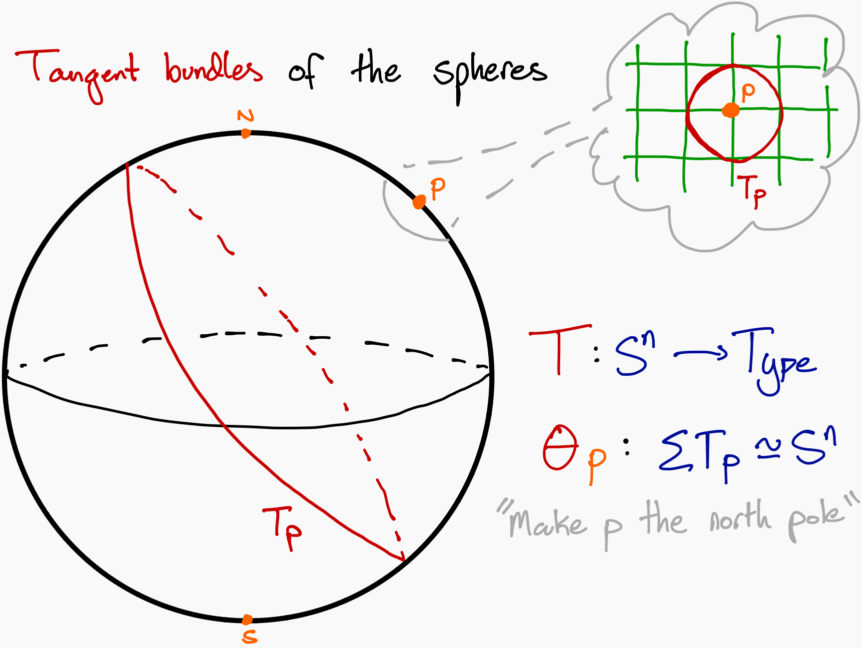
- Tangent bundles of the spheres





$$T : S^n \to \overset{\text{Sphere}}{\cancel{\text{Type}}}$$

- Tangent bundles of the spheres



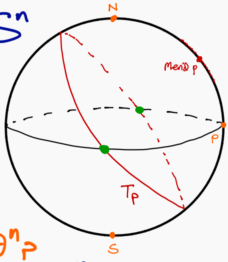
$$T : S^n \to \mathrm{Type}$$

$$\Theta_p : \Sigma T_p \simeq S^n$$

"Make p the north pole"

- Tangent bundles of the spheres

- Inductively define $T^n : S^n \to \mathrm{Type}$ and
  $\Theta^n : (p : S^n) \to \Sigma T^n_p \simeq S^n$



- Case $n \equiv -1$ is trivial.
- $T^{n+1} : \Sigma S^n \to \mathrm{Type}$

  $N \longmapsto S^n$ the equator

  $S \longmapsto S^n$

  $p : S^n \longmapsto S^n \overset{(\Theta^n_p)^{-1}}{=\!=} \Sigma T^n \overset{\mathrm{flip}}{=\!=} \Sigma T^n \overset{\Theta^n_p}{=\!=} S^n$

- $\Theta^{n+1} : (p : \Sigma S^n) \to \Sigma T^{n+1} \simeq \Sigma S^n$

  $N \longmapsto \Sigma S^n \xrightarrow{\mathrm{id}} \Sigma S^n$

  $S \longmapsto \Sigma S^n \xrightarrow{\mathrm{flip}} \Sigma S^n$

  $p : S^n \longmapsto$

$$\begin{tikzcd}
\Sigma\Sigma T^n \arrow[d, "\Sigma\mathrm{flip}"'] \arrow[d, bend left, "\mathrm{flip}"] \arrow[d, phantom, "\mathrm{Lem.}"] & \Sigma S^n \arrow[l, "\Sigma(\Theta^n_p)^{-1}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "\mathrm{nat.}"] & \\
\Sigma\Sigma T^n \arrow[r, "\Sigma\Theta^n_p"'] & \Sigma S^n \arrow[r, "\mathrm{flip}"'] & \Sigma S^n
\end{tikzcd}$$

- Crucial lemmas about $\mathrm{flip} : \Sigma A \xrightarrow{\sim} \Sigma A$

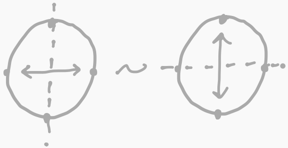
$$\begin{aligned} N &\longmapsto S \\ S &\longmapsto N \\ \mathrm{merid}\ p &\longmapsto (\mathrm{merid}\ p)^{-1} \end{aligned}$$

- Naturality:

$$\begin{tikzcd} \Sigma A \arrow[r, "\Sigma f"] \arrow[d, "\mathrm{flip}"'] \arrow[dr, phantom, "//"] & \Sigma B \arrow[d, "\mathrm{flip}"] \\ \Sigma A \arrow[r, "\Sigma f"'] & \Sigma B \end{tikzcd}$$

- Lem:

$$\begin{tikzcd} \Sigma\Sigma A \arrow[r, bend left, "\Sigma \mathrm{flip}"] \arrow[r, bend right, "\mathrm{flip}"'] \arrow[r, phantom, "\|"] & \Sigma\Sigma A \end{tikzcd}$$



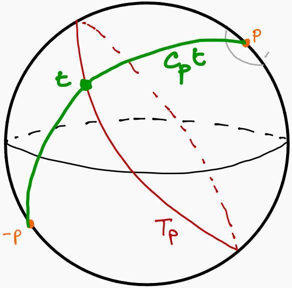
- involution: $\mathrm{flip} \circ \mathrm{flip} = \mathrm{id}$

The antipode $a^n : S^n \xrightarrow{\sim} S^n$ is defined inductively by

$$a^{n+1} \equiv \mathrm{flip} \circ \Sigma a^n$$

So:

$$a^n = \mathrm{flip} \circ \Sigma \mathrm{flip} \circ \Sigma^2 \mathrm{flip} \cdots \Sigma^n \mathrm{flip} = \begin{cases} \mathrm{flip} & \text{if } n \text{ even} \\ \mathrm{id} & \text{if } n \text{ odd} \end{cases}$$



We can show that $\Theta\, p\, N = p$ and $\Theta\, p\, S = a\, p$

Therefore, we have a map:

$$\begin{tikzcd} T_p \arrow[r, "\mathrm{merid}"] \arrow[rrr, bend left, "c_p"] & (N =_{\Sigma T_p} S) \arrow[r, "\mathrm{ap}\,\Theta\, p"] & (\Theta\, p\, N = \Theta\, p\, S) \arrow[r, phantom, "\simeq"] & (p = a\, p) \end{tikzcd}$$

So, from a section $s : (p : S^n) \to T_p$, we get a homotopy $\hat{s} : id = a$

But this can only happen if n is odd!

Define $\mathrm{BAut}^+(X) :\equiv (Y:\mathrm{Type}) \times \| Y = X \|_0$

We have $T : S^n \to \mathrm{BAut}^+(S^{n-1})$.

oriented spheres

Define the Euler class:

$$e : \mathrm{BAut}^+(S^n) \longrightarrow \mathrm{BAut}^+(K(\mathbb{Z},n))$$
$$(S, \ldots) \longmapsto (\|S\|_n, \ldots)$$

The theory of banded types (Tsaxeras, et al.) proves

$$\mathrm{BAut}^+(K(\mathbb{Z},n)) \xrightarrow{\sim} K(\mathbb{Z},n+1) \quad \text{delooping}$$
$$\mathrm{Aut}^+(K(\mathbb{Z},n)) \xrightarrow{ev_{pt}} K(\mathbb{Z},n)$$

So $|e \circ T|_0$ is a class in $H^n(S^n)$.

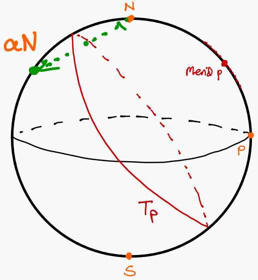
To compute $e\circ T^{\mathrm{adj}}$, transpose

$$\Sigma S^n \xrightarrow{T} BAut^+(S^n) \xrightarrow{e} BAut^+(K(\mathbb{Z},n))$$

to:

$$p \longmapsto \left(S^n \xrightarrow{\Theta_p^{-1}} \Sigma T_p \xrightarrow{\text{flip}} \Sigma T_p \xrightarrow{\Theta_p} S^n\right)$$

$$\begin{tikzcd}
S^n \arrow[r, "\text{mer'd}"] \arrow[rrd, dashed, "\alpha"'] & (N \underset{S^{n+1}}{=} S) \arrow[r, "T_*"] & Aut^+(S^n) \arrow[r, "\|\text{-}\|_n"] \arrow[d, "ev_N"'] & Aut^+(K(\mathbb{Z},n)) \arrow[d, "ev_{pt}"] \\
& & S^n \arrow[r, "|\cdot|_n"] & K(\mathbb{Z},n)
\end{tikzcd}$$



i.e.: $e\circ T = \deg \alpha$.

We then compute that

$$\alpha_*(\text{mer'd}\,x) = \text{mer'd}(ax)^{\wedge}\cdot \text{mer'd}(x)$$

Therefore, $e\circ T = a + \mathrm{id}$ in $H^n(S^n)$

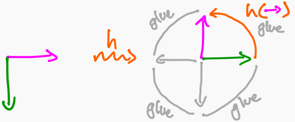
We have a more general approach using joins:

Def: A **reflection** $r : A \simeq A$ satisfies

① **reflect** : $\begin{tikzcd} A * A \arrow[r, bend left, "r * A"] \arrow[r, bend right, "\mathrm{swap}"'] \arrow[r, phantom, "||"] & A * A \end{tikzcd}$

$t \overset{h}{\mapsto} \begin{bmatrix} \cos\frac{\pi t}{2} & -\sin\frac{\pi t}{2} \\ \sin\frac{\pi t}{2} & \cos\frac{\pi t}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

② **coh** : $\mathrm{reflect}(\mathrm{inl}(a)) = \mathrm{glue}(a, a)$



Def: An **A-orthorsor** is

① $\tau : E \to \mathrm{Type}$ ② $\theta : (e : E) \to (A * \tau(e) \simeq E)$

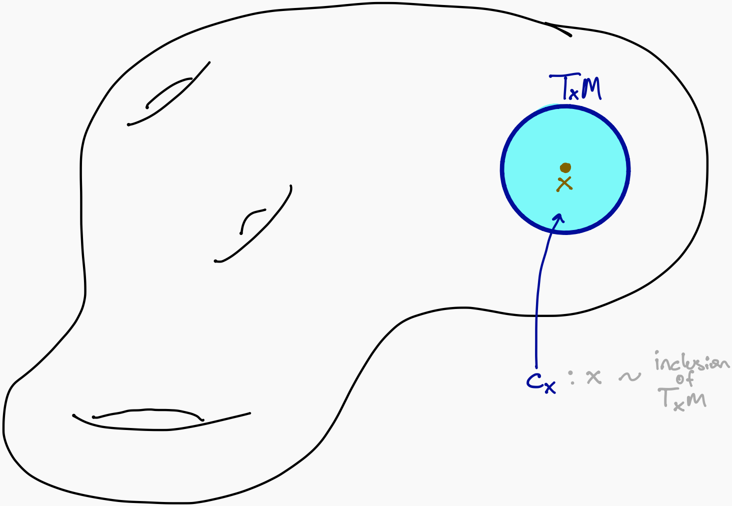
"orthogonal complement"

Construction: A **join** for orthorsors:

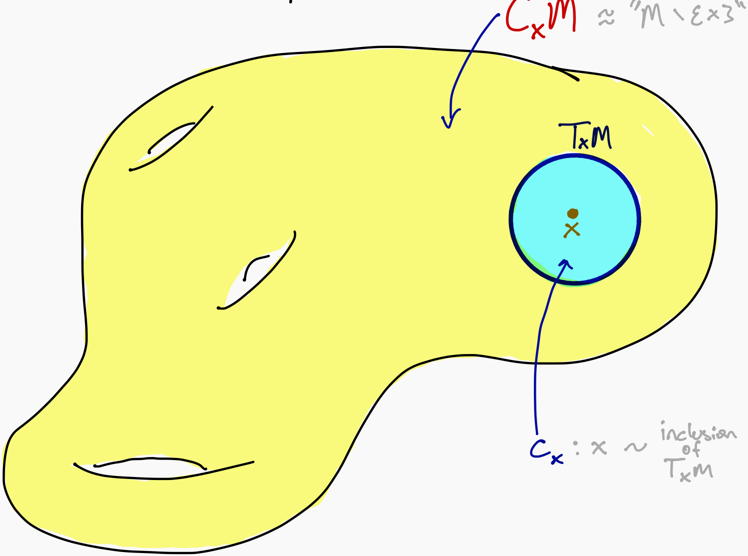
$(E, \tau^E, \theta^E) * (F, \tau^F, \theta^F) :\equiv (E * F, \text{"}\tau^E * F \leadsto E * \tau^F\text{"}, \ldots)$

Tangents arise as $(A, \emptyset, \mathrm{const\ id})^{*n} : \mathrm{Orthorsor}(A)$

What makes a sphere bundle a "tangent bundle"?



What makes a sphere bundle a "tangent bundle"?



Def: A homotopy pre-manifold structure on a type M consists of a pushout complement

$$
\begin{tikzcd}
T_pM \arrow[r, "\partial_p"] \arrow[d, "!"'] & C_pM \arrow[d, "?_p"] \\
* \arrow[r, "p"'] \arrow[ur, phantom, "c_p\ \Nearrow"] \arrow[ur, phantom, "\ulcorner", very near end] & M
\end{tikzcd}
$$

≈ Lashoff '72, classically

of every point $p : M$, where T_pM is merely a sphere.

Thm:

$$
\begin{tikzcd}
T_p \arrow[r, "!"] \arrow[d, "!"'] & * \arrow[d, "\mathrm{ap}"] \\
* \arrow[r, "p"'] \arrow[ur, phantom, "c_p\ \Nearrow"] \arrow[ur, phantom, "\ulcorner", very near end] & S^n
\end{tikzcd}
$$

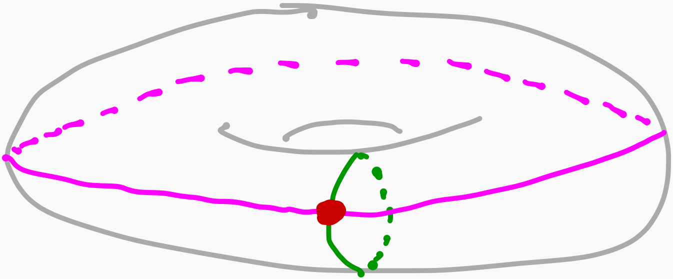
endows S^n with the structure of a homotopy pre-manifold.

Suppose we have homotopy pre-manifolds M and N

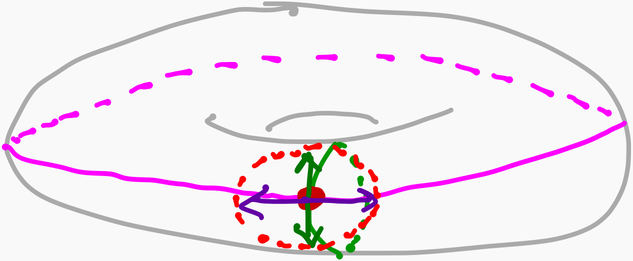
$$
\begin{tikzcd}
T_pM \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & C_pM \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\
* \arrow[r, "p"'] & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T_qN \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & C_qN \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\
* \arrow[r, "q"'] & N
\end{tikzcd}
$$

How can we define a pre-manifold structure on $M \times N$?

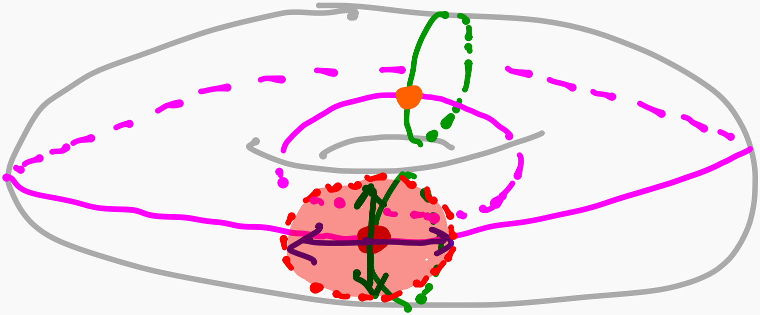
$$
\begin{tikzcd}
? \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & ? \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\
* \arrow[r, "{(p,q)}"'] & M \times N
\end{tikzcd}
$$



$$* \xrightarrow[(p,q)]{} S^1 \times S^1$$



$$\begin{tikzcd} T_pS^1 \times T_qS^1 \arrow[d] & \\ * \arrow[r, "(p,q)"'] & S^1 \times S^1 \end{tikzcd}$$



$$\begin{tikzcd}
T_pS^1 * T_qS^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & S^1 \vee_{aq\,ap} S^1 \arrow[d] \\
* \arrow[r, "(p,q)"'] & S^1 \times S^1
\end{tikzcd}$$

This is a case of the pushout product:

$$\begin{array}{c} A_1 \\ {\scriptstyle f}\downarrow \\ B_1 \end{array}, \quad \begin{array}{c} A_2 \\ {\scriptstyle g}\downarrow \\ B_2 \end{array} \quad \longmapsto \quad \begin{tikzcd} A_1 \times A_2 \arrow[r, "f \times id"] \arrow[d, "id \times g"'] & B_1 \times A_2 \arrow[d] \arrow[ddr, bend left, "id \times g"] & \\ A_1 \times B_2 \arrow[r] \arrow[drr, bend right, "f \times id"'] & f \,\square\, g \arrow[dr, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & B_1 \times B_2 \end{tikzcd}$$

E.g.:

Join:

$$\begin{array}{c} A_1 \\ {\scriptstyle !}\downarrow \\ * \end{array}, \quad \begin{array}{c} A_2 \\ {\scriptstyle !}\downarrow \\ * \end{array} \quad \longmapsto \quad \begin{array}{c} A_1 * A_2 \\ {\scriptstyle !}\downarrow \\ * \end{array}$$

Wedge:

$$\begin{array}{c} * \\ {\scriptstyle b_1}\downarrow \\ B_1 \end{array}, \quad \begin{array}{c} * \\ {\scriptstyle b_2}\downarrow \\ B_2 \end{array} \quad \longmapsto \quad \begin{array}{c} B_1 \vee B_2 \\ \downarrow \\ B_1 \times B_2 \end{array}$$

Suppose we have homotopy pre-manifolds M and N

$$\begin{tikzcd} T_pM \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & C_pM \arrow[d, "\eta_p^M"] \\ * \arrow[r, "p"'] & M \end{tikzcd} \qquad \begin{tikzcd} T_qN \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & C_qN \arrow[d, "\eta_q^N"] \\ * \arrow[r, "q"'] & N \end{tikzcd}$$

To define a pre-manifold structure on $M \times N$,

Take the pushout product of the constituent squares:

$$\begin{tikzcd} T_pM * T_qN \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & \eta_p^M \square \eta_q^N \arrow[d] \\ * \arrow[r, "(p,q)"'] & M \times N \end{tikzcd}$$

For this to work, we need the following lemma:

Given pushout squares

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"'] \arrow[dr, phantom, "S_1"] & B_1 \arrow[d, "g_1"] \\ C_1 \arrow[r] & D_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[r] \arrow[d, "f_2"'] \arrow[dr, phantom, "S_2"] & B_2 \arrow[d, "g_2"] \\ C_2 \arrow[r] & D_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Their pushout product is a pushout too.

$$\begin{tikzcd} f_1 \square f_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "S_1 \square S_2"] & g_1 \square g_2 \arrow[d] \\ C_1 \times C_2 \arrow[r] & D_1 \times D_2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Some

Future Work

- Thom class vs Euler Class
- Stable normal bundles and Atiyah Duality

Thanks!